

Fast decay of correlations of equilibrium states of open classes of non-uniformly expanding maps and potentials

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Abstract

We study the existence, uniqueness and rate of decay of correlation of equilibrium measures associated to robust classes of non-uniformly expanding local diffeomorphisms and Hölder continuous potentials. The approach used in this paper is the spectral analysis of the Ruelle-Perron-Frobenius transfer operator. More precisely, we combine the expanding features of the eigenmeasures of the transfer operator with a Lasota-Yorke type inequality to prove the existence of an unique equilibrium measure with fast decay of correlations.

1 Introduction

The thermodynamical formalism of uniformly expanding local diffeomorphisms developed by Ruelle, Sinai, Bowen (among others) in the seventies is a major example of applications of ideas of statistical mechanics into the ergodic theory of dynamical systems. Indeed, the dictionary between one-dimensional lattices and expanding systems (in particular, Gibbs distributions and equilibrium measures) established via Markov partitions permits to translate results in statistical mechanics into relevant theorems in the ergodic theory of uniformly expanding systems.

However, the extension of this theory beyond the uniformly expanding context reveals fundamental difficulties, even if we are restricted to the non-uniformly expanding realm (i.e., abundance of positive Lyapounov exponents). In fact, when dealing with such problem, a major trouble is the absence of generating finite Markov partitions: in general, being very optimistic, one can expect only Markov partitions with infinitely many symbols, leading us to consider the thermodynamical formalism of gases with infinitely many states, a hard subject not yet completely understood (see e.g. [BS] for a recent progress).

On the other hand, some substantial advance was made by Alves, Bonatti and Viana [ABV] concerning the existence and uniqueness of SRB measures

for some robust classes of non-uniformly expanding maps: they essentially assume that *Lebesgue* almost every point has only non-zero Lyapounov exponents. Since the physical measures should be absolutely continuous with respect to the Lebesgue measure, this condition really says something about the problem of existence of SRB measures. But, it is not clear how to formulate the correct analog of this assumption in the case of equilibrium states with respect to arbitrary potentials (because, in general, equilibrium measures are not absolutely continuous with respect to Lebesgue). In this direction, some recent contributions have been given by several authors: see, e.g., the work of Bruin and Keller [BK] for interval maps, Denker and Urbanski [DU], [U] for rational functions of the Riemann sphere, Wang and Young [WY] for Hénon-like maps; Buzzi, Maume-Deschamps, Sarig and Yuri [Bu], [BMD], [BS], [S], [Yu] for countable Markov shifts and piecewise expanding maps. Many of these papers deal with dynamical systems with neutral periodic points, a relevant source of example of non-hyperbolic systems.

Nevertheless, we consider a different open class \mathcal{C} of non-uniformly expanding maps obtained through *pitch-fork bifurcation of a periodic point of an expanding linear torus endomorphism*. Here, the maps are assumed to be uniformly expanding outside a small neighborhood of the periodic point where the pitch-fork bifurcation was made (but possibly of saddle type near the periodic point) and the potential ϕ is supposed to have its oscillation $\max \phi - \min \phi$ not too large. In this setting, Oliveira [O] proved the existence of ergodic equilibrium states whose Lyapounov exponents are positive. Also, Arbieto, Matheus and Oliveira [AMO] extended Oliveira's approach to obtain similar results for small random perturbations of these maps.

Recently, Oliveira and Viana [OV1] improved the results of [O] when the potential ϕ is constant by showing that the equilibrium state with respect to this potential (i.e., the measure of maximal entropy) is unique. The idea is to use the *Ruelle-Perron-Frobenius transfer operator* to construct a maximal measure. Then, the uniqueness result follows from the good spectral properties of Ruelle's operator. We would like to point out that Oliveira and Viana announced in the preprint [OV2] that this method can be pushed a little bit more to extend the results of [OV1] for non-constant potentials with low oscillation.

The present paper relies on the basic strategy of [OV2], i.e., we analyze the spectral properties of the transfer operator obtaining our main result:

"Given $f \in \mathcal{C}$ and ϕ is an Hölder continuous potential with $\max \phi - \min \phi$ sufficiently small, there exists a unique equilibrium measure μ with respect to (f, ϕ) . Also, the Lyapounov exponents of μ are positive and μ has exponential decay of correlations."

The organization of this paper is as follows. In section 2, we introduce the open classes of non-uniformly expanding maps and Hölder continuous potentials (with low variation) treated here. In section 3, we consider the Ruelle-Perron-Frobenius transfer operator \mathcal{L}_ϕ and the eigenmeasures ν of its dual operator \mathcal{L}_ϕ^* . After that, we prove that ν is expanding (i.e., hyperbolic times are abundant at ν -generic points) and \mathcal{L}_ϕ satisfies a Lasota-Yorke type inequality. Next, we show that this ensures the existence of an eigenfunction h of \mathcal{L}_ϕ . As a consequence,

we can obtain the central result of the section 3: the existence of an unique f -invariant probability μ which is absolutely continuous with respect to ν ($=$ an eigenmeasure of \mathcal{L}_ϕ^*); moreover, μ has exponential decay of correlations. In section 4, we prove that the f -invariant probability μ is an equilibrium state of (f, ϕ) and any equilibrium measure η verifies that $\frac{1}{h}\eta$ is an eigenmeasure of \mathcal{L}_ϕ^* . In section 5, we complete the proof of our main result by showing the uniqueness of equilibrium measure μ . The idea is that the expanding features of any eigenmeasure of \mathcal{L}_ϕ^* implies that they are all equivalent so that $\frac{1}{h}\eta$ is equivalent to ν from the results of section 4. Then, by ergodicity, it follows that $\eta = h\nu = \mu$, as desired. Finally, in section 6, we include some ergodic properties of the equilibrium states constructed here: spectral gap, stochastic stability and large deviations. The proof of these results are simple modifications of their analogous in the uniformly expanding context (based on the lemmas proved in this article) and they were included only to illustrate the power of the machinery developed in the previous sections. To make the exposition more self-contained, we included also three appendices containing some facts used along the proof of the main result (but not showed in the text in order to do not interrupt the argument).

To close the introduction, we would like to say that this article is only starts a program to understand the ergodic theory of non hyperbolic systems. Indeed, some questions motivated by our theorems are:

- Is it possible to extend our existence, uniqueness and decay of correlations results for more general potentials?
- What about analogous theorem for robust classes of *non-uniformly hyperbolic diffeomorphisms* (e.g., pitch-fork bifurcation of the expanding direction of a periodic saddle-point of an Anosov diffeomorphism) and/or *non-uniformly expanding local diffeomorphisms with critical points* (e.g., Viana maps)?

2 Preliminaries

Let $f : M \rightarrow M$ be a continuous transformation on a compact space M and $\phi : M \rightarrow \mathbb{R}$ be a continuous function. An f -invariant measure is an *equilibrium state* of f for the potential ϕ if it maximizes the functional

$$\eta \mapsto h_\eta(f) + \int \phi d\eta,$$

among all f -invariant probabilities η .

2.1 The class of transformations and potentials

We consider $f : M \rightarrow M$ is a C^1 local diffeomorphism on a compact boundary-less manifold M such that

(H1) There exists connected closed sets (with finite inner diameter)

$$\mathcal{R} = \{R_1, \dots, R_q, R_{q+1}, \dots, R_{p+q}\}$$

whose interiors are pairwise disjoint such that $\text{diam}(\mathcal{R}) < \varepsilon$ for some $\varepsilon > 0$, $\bigcup_i R_i = M$, f is injective on each R_i and, for some constants $\sigma_1 < 1$ and $\delta_0 > 0$,

- f is expanding at every $x \in R_{q+1} \cup \dots \cup R_{p+q}$: $\|Df(x)^{-1}\| \leq \sigma_1^{-1}$;
- f is never too contracting: $\|Df(x)^{-1}\| \leq 1 + \delta_0$ for every $x \in M$;
- \mathcal{R} is a transitive Markov partition: the image $f(R_i)$ of every atom is the union of some atoms R_j and there exists N such that $f^N(R_i) = M$, $i = 1 \dots, p + q$;

It is worth to point out that the partition \mathcal{R} is *not required to be generating*, i.e., the diameters of the cylinders of length n do not need to decrease to zero when $n \rightarrow \infty$.

We denote by k the number of pre-images of any point x under f . This number does not depend on x because f is a local diffeomorphism. We assume that $k > q$ (i.e., every point has some pre-image in a good rectangle R_i , $i = q + 1 \dots, p + q$). Moreover, the potential $\phi : M \rightarrow \mathbb{R}$ is supposed to verify

(H2) ϕ is α -Hölder-continuous and its variation $\max \phi - \min \phi$ is sufficiently small, i.e.,

$$\max \phi - \min \phi < \epsilon_0(f).$$

Remark 2.1. By the definition of equilibrium states, it is not difficult to see that any equilibrium state η of ϕ is also an equilibrium state of the potential $\phi + c$, for every constant $c \in \mathbb{R}$. So, up to consider $\phi - \inf \phi$ instead of ϕ , we can assume that $\inf \phi = 0$ and, in particular, $\phi \geq 0$.

2.2 Statement of the main theorems

The main theorems of this paper concerns the existence, uniqueness, fast mixing (exponential decay of correlations) and the oscillation of the Birkhoff sums of Hölder observables around the expected value (central limit theorem). Unfortunately, it turns out that our techniques only can be applied if some extra control of the contraction of the derivative of f is assumed.

In this article, we deal with two different candidates to be the extra condition, both of them ensuring the necessary control of the contraction for the application of our method.

The first candidate is:

(H3) $\max_{x \in M} \log \|\Lambda^{d-1} Df(x)\| < \log k$, where d is the dimension of M and Λ^k is the k -th exterior power of Df .

Morally speaking, (H3) means that the $(d - 1)$ -dimensional volume is not expanded too much.

Our first main result is related with the ergodic theory of equilibrium states of the class of transformations verifying (H1), (H2) and (H3):

Theorem A. *Under the hypothesis (H1), (H2) and (H3) above, if δ_0 is sufficiently small, then f has an unique equilibrium state μ for the potential ϕ . Moreover, μ has exponential decay of correlations for Hölder observables: there exists some constants $0 < \tau < 1$ and $K > 0$ such that, for all $u \in L^1(\nu)$, $v \in C^\alpha(M)$ and $n \geq 0$,*

$$\left| \int_M (u \circ f^n) v d\mu - \int_M u d\mu \int_M v d\mu \right| \leq K(u, v) \cdot \tau^n,$$

and the central limit theorem property: for all u Hölder continuous, the random variable

$$\frac{1}{\sqrt{n}} \sum_{j=0}^{n-1} (u \circ f^j - \int_M u d\mu)$$

converges to a Gaussian law. More precisely, let u be a α -Hölder continuous function and

$$\sigma^2 := \int v^2 d\mu + 2 \sum_{j=1}^{\infty} v \cdot (v \circ f^j) d\mu, \quad \text{where} \quad v = u - \int u d\mu.$$

Then $\sigma < \infty$ and $\sigma = 0$ iff $u = \varphi \circ f - \varphi$ for some $\varphi \in L^1(\mu)$. If $\sigma > 0$, then, for every interval $A \subset \mathbb{R}$,

$$\mu \left(x \in M : \frac{1}{\sqrt{n}} \sum_{j=0}^{n-1} \left(u(f^j(x)) - \int u d\mu \right) \in A \right) \rightarrow \frac{1}{\sigma \sqrt{2\pi}} \int_A e^{-\frac{t^2}{2\sigma^2}} dt,$$

as $n \rightarrow \infty$.

The second candidate is:

(H4) $|\det Df(x)| \geq \sigma_2 > q$ for all $x \in M$ (i.e., f is volume-expanding everywhere) and there exists a set $W \subset R_1 \cup \dots \cup R_q$ containing $V := \{x \in M : \|Df(x)^{-1}\| > \sigma_1^{-1}\}$ such that

$$M_1 > m_2 \quad \text{and} \quad m_2 - m_1 < \beta,$$

where m_1, m_2 are the infimum and the supremum of $\log \|\det Df\|$ on V , resp. and M_1, M_2 are the infimum and the supremum of $\log \|\det Df\|$ on $M - W$, resp.

Informally, (H4) says that f is volume-expanding everywhere, i.e., f can not contract all the directions at the same time, so that the basins of sinks are not

allowed as a region where f can eventually contract. Also, the expansion of volume near the region of Df -contraction is close to be constant and, furthermore, the volume is expanded more strongly outside the region of contraction than inside of it.

Our second main result is related with the ergodic theory of equilibrium states of the class of transformations verifying (H1), (H2) and (H4):

Theorem B. *Under the hypothesis (H1), (H2) and (H4) above, if β and δ_0 are sufficiently small, then f has an unique equilibrium state μ for the potential ϕ . Moreover, μ has exponential decay of correlations for Hölder observables: there exists some constants $0 < \tau < 1$ and $K > 0$ such that, for all $u \in L^1(\nu)$, $v \in C^\alpha(M)$ and $n \geq 0$,*

$$\left| \int_M (u \circ f^n) v d\mu - \int_M u d\mu \int_M v d\mu \right| \leq K(u, v) \cdot \tau^n,$$

and the central limit theorem property: for all u Hölder continuous, the random variable

$$\frac{1}{\sqrt{n}} \sum_{j=0}^{n-1} (u \circ f^j - \int_M u d\mu)$$

converges to a Gaussian law. More precisely, let u be a α -Hölder continuous function and

$$\sigma^2 := \int v^2 d\mu + 2 \sum_{j=1}^{\infty} v \cdot (v \circ f^j) d\mu, \quad \text{where} \quad v = u - \int u d\mu.$$

Then $\sigma < \infty$ and $\sigma = 0$ iff $u = \varphi \circ f - \varphi$ for some $\varphi \in L^1(\mu)$. If $\sigma > 0$, then, for every interval $A \subset \mathbb{R}$,

$$\mu \left(x \in M : \frac{1}{\sqrt{n}} \sum_{j=0}^{n-1} \left(u(f^j(x)) - \int u d\mu \right) \in A \right) \rightarrow \frac{1}{\sigma \sqrt{2\pi}} \int_A e^{-\frac{t^2}{2\sigma^2}} dt,$$

as $n \rightarrow \infty$.

Let us comment about our conditions on the constants δ_0, β and $\epsilon_0(f)$. We start with three conditions which should hold in the context of both theorems:

Most of the arguments of this paper strongly relies on the non-uniform expansion property. A natural way to find it is to count the number of itineraries with frequent visit to $\mathcal{B} = R_1 \cup \dots, R_q$. Before doing this, we recall the following lemma (which is a direct corollary of Stirling's formula).

Lemma 2.2. *Given $\gamma \in (0, 1)$, let $I_{\gamma, n}$ be*

$$\{(i_0, \dots, i_{n-1}) \in \{1, \dots, q, q+1, \dots, p+q\}^n; \#\{0 \leq j < n; i_j \leq q\} > \gamma n\}$$

and $c_\gamma = \limsup_{n \rightarrow \infty} \frac{1}{n} \log \#I_{\gamma, n}$. Then, $c_\gamma \rightarrow \log q$ when $\gamma \rightarrow 1$.

Proof. See the appendix I for a proof. \square

We fix some γ close to 1 such that $c_\gamma < \log k$ (which is possible since $q < k$) and take δ_0 small depending on σ_1 and γ satisfying

$$(1 + \delta_0)^\gamma \sigma_1^{-(1-\gamma)} \leq e^{-4c} < 1, \quad (1)$$

for some constant $c > 0$.¹

Denote $c_0(f) = c_\gamma$ and consider $\epsilon_0(f)$ such that²

$$\epsilon_0(f) < \log k - c_0(f). \quad (2)$$

Also we assume that $\epsilon_0(f)$ verifies

$$e^{\epsilon_0(f)} \cdot \left(\frac{k-q}{k} \sigma_1^{-\alpha} + \frac{q}{k} (1 + \delta_0)^\alpha \right) < 1 \quad (3)$$

Clearly it is satisfied if $\epsilon_0(f)$ is small depending only on σ_1, k, q, α , since $k > q$.³

Now we turn to the specific conditions of the main theorems:

- *Precise conditions on the constants in theorem A.* Here, the hypothesis (H1), (H2), (H3) and the restrictions (1), (2), (3) are *almost* sufficient and, in fact, it is necessary only to complement the restriction (2) with

$$\epsilon_0(f) < \log k - \max_{x \in M} \log \|\Lambda^{d-1} Df(x)\|. \quad (4)$$

- *Precise conditions on the constants in theorem B.* Besides the hypothesis (H1), (H2), (H4) and the restrictions (1), (2), (3) above, we need two more conditions. First of all, in the appendix of [ABV], Alves, Bonatti and Viana proved that there exists some $\gamma_0 > 0$ depending only on σ_2, p, q such that the orbit of Lebesgue almost every point $x \in M$ spends a fraction γ_0 of the time in $R_{q+1} \cup \dots \cup R_{p+q}$:

$$\frac{1}{n} \# \{0 \leq j < n; f^j(x) \in R_{q+1} \cup \dots \cup R_{p+q}\} \leq \gamma_0 n,$$

for every large n .

We take $\gamma_0 < \gamma < 1$ close to 1 such that

$$\gamma m_2 + (1 - \gamma) M_2 < \gamma_0 m_1 + (1 - \gamma_0) M_1 - d \log(1 + \delta_0), \quad (5)$$

where d is the dimension of M . Note that by the hypothesis (H4), $m_2 < M_1$ and $m_2 - m_1 < \beta$. Hence, this condition can be verified if δ_0, β are

¹This condition ensures that if the orbit of a point does not frequently visit \mathcal{B} , then we have non-uniform expansion along it.

²Roughly speaking, (2) forces the orbit of generic points of certain measures (e.g., equilibrium states) to stay a large portion of time outside \mathcal{B} . See lemma 3.2.

³The role of (3) is to guarantee that the equilibrium state obtained by our construction has exponential decay of correlations and the central limit theorem property.

sufficiently small (depending only on γ_0 , i.e., σ_2, p, q , and $M_1 - m_2$). In particular, our condition “ $1 - \gamma$ is small” above depends only on σ_2, p, q and $M_1 - m_2$.

So, we get that β is small depending only on σ_2, p, q , $M_1 - m_2$ and δ_0 is small depending on σ_1, σ_2, p, q , $M_1 - m_2$.

Now we turn our attention to what does “ $\epsilon_0(f)$ is small” in hypothesis (H2) means. We take any $0 < \rho$ such that

$$\rho \cdot (\gamma_0 m_1 + (1 - \gamma_0)M_1 - d \log(1 + \delta_0)) < \gamma m_2 + (1 - \gamma)M_2.$$

Note that $\rho < 1$ by (5). The last condition on $\epsilon_0(f)$ is

$$\epsilon_0(f) < (1 - \rho)h_{top}(f). \quad (6)$$

Finally, in the case of theorem B, we impose all the conditions above on the parameters β, δ_0 and $\epsilon_0(f)$.

For future reference, we recall that Oliveira proved

Theorem 2.3 (Oliveira [O]). *In the context of the theorem B, i.e., under our hypothesis (H1), (H2), (H4), and if the previous conditions on β, δ_0 and $\epsilon_0(f)$ holds, there are (ergodic) equilibrium states of f for the potential ϕ . Moreover, any ergodic equilibrium state η of f with respect to ϕ satisfies $\eta(V) \leq \gamma$. Furthermore, the Lyapounov exponents of η are all positive and η has infinitely many hyperbolic times (see definition 3.3 below).*

After this preparation, we are ready to study the Ruelle-Perron-Frobenius transfer operator and its spectral properties.

3 The Ruelle-Perron-Frobenius operator

The Ruelle-Perron-Frobenius transfer operator $\mathcal{L}_\phi : C(M) \rightarrow C(M)$ of $f : M \rightarrow M$ is defined on the space $C(M)$ of continuous functions $g : M \rightarrow \mathbb{R}$ by

$$\mathcal{L}_\phi g(x) = \sum_{f(y)=x} e^{\phi(y)} g(y).$$

Denote by λ the spectral radius of \mathcal{L}_ϕ . Let ν be an eigenmeasure associated to the eigenvalue λ of the dual operator \mathcal{L}_ϕ^* (i.e., $\mathcal{L}_\phi^*(\nu) = \lambda \cdot \nu$). In particular, $\lambda = \int \mathcal{L}_\phi 1 d\nu$ and, hence, $k \leq \lambda \leq k \cdot e^{\max \phi}$.

Note that the eigenmeasure ν is not necessarily f -invariant. However, we can capture its dynamical features (with respect to f) through the concept of Jacobian of a measure μ with respect to f : it is the (essentially) unique function $J_\mu f$ such that

$$\mu(f(A)) = \int_A J_\mu f,$$

for any measurable set A where $f|_A$ is injective. Equivalently, $J_\mu f = \frac{d(f_*\mu)}{d\mu}$. In general, Jacobians do not exist but, if f is countable to one then $J_\mu f$ does exist for any measure μ .

As a first job, let us compute the Jacobian of the eigenmeasure ν :

Lemma 3.1. *If $\mathcal{L}_\phi^* \nu = \lambda \nu$, then $J_\nu f = \lambda e^{-\phi}$.*

Proof. Let A be any measurable set such that $f|_A$ is injective. Take a sequence g_n of continuous functions satisfying $g_n \rightarrow \chi_A$ ν -a.e. and $\sup |g_n| \leq 2$ for all n . By definition,

$$\mathcal{L}_\phi(e^{-\phi} g_n) = \sum_{f(y)=x} g_n(y).$$

Since the right-hand of this equation is ν -a.e. converging to $\chi_{f(A)}(x)$, applying the dominated convergence theorem and the hypothesis $\mathcal{L}_\phi^* \nu = \lambda \nu$, we get

$$\int \lambda e^{-\phi} g_n d\nu = \int e^{-\phi} g_n d(\mathcal{L}_\phi^* \nu) = \int \mathcal{L}_\phi(e^{-\phi} g_n) d\nu \rightarrow \nu(f(A)).$$

However, the left-hand of the last equation converges to $\int_A \lambda e^{-\phi} d\nu$. Thus,

$$\nu(f(A)) = \int_A \lambda e^{-\phi} d\nu.$$

In other words, $J_\nu f = \lambda e^{-\phi}$. □

Next, we will show how the explicit formula for the Jacobian of ν can be used to get some non-trivial information about the dynamics of f -orbits of ν generic points.

3.1 Non-uniform expansion and its consequences

Standing hypothesis of this subsection. For the proof of the results of the present subsection, we assume only (H1), (H2) and the conditions (1), (2).

Note that (2) and lemma 3.1 lead us to

$$J_\nu f \geq \lambda e^{-\max \phi} \geq e^{\log k - \max \phi} > e^{c_0(f)}.$$

Hence, we can find some $\kappa > c_0(f)$ such that

$$J_\nu f > e^\kappa > q. \tag{7}$$

From this lower bound on $J_\nu f$, we will derive a relevant combinatorial result about the visit of ν -generic f -orbits to the bad region \mathcal{B} of eventual contraction:

Lemma 3.2. *Any probability measure ν such that $\mathcal{L}_\phi^* \nu = \lambda \nu$ satisfies*

$$\nu(G) = 1,$$

where $G := \left\{ x \in M : \limsup_{n \rightarrow \infty} \frac{1}{n} \# \{ 0 \leq j \leq n; f^j(x) \in \mathcal{B} \} \leq \gamma \right\}$.

Proof. Consider any cylinder $R^n \in \mathcal{R}^n$. Since f is injective on each element of \mathcal{R} , it follows that f^n is injective on each element of \mathcal{R}^n . Hence, by the definition of Jacobian and the inequality (7),

$$1 \geq \nu(f^n(R^n)) = \int_{R^n} J_\nu f^n d\nu = \int_{R^n} \prod_{j=0}^{n-1} (J_\nu f \circ f^j) d\nu \geq e^{\kappa n} \nu(R^n).$$

In particular $\nu(R^n) \leq e^{-\kappa n}$.

Let $A(n)$ be the union of all $R^n \in I_{\gamma,n}$, that is, all n -cylinders $R^n = [i_0, \dots, i_{n-1}]$ such that

$$\frac{1}{n} \#\{0 \leq j < n; i_j \leq q\} > \gamma.$$

Recall the terminology of the lemma 2.2. The cardinality of $I_{\gamma,n}$ grows with n at the exponential rate c_γ . Note that we have $\kappa > c_\gamma$. So, for some $C > 0$,

$$\nu(A(n)) = \sum_{R^n \in I_{\gamma,n}} \nu(R^n) \leq C e^{c_\gamma n} e^{-\kappa n},$$

i.e., $\nu(A(n)) \rightarrow 0$ exponentially fast. Then, by Borel-Cantelli lemma, for every x in a full ν -measure set and for every n large enough, we have $x \notin A(n)$. This finishes the proof, since $x \notin A(n)$ means exactly that $R^n(x) = [i_0, \dots, i_{n-1}]$ satisfies $\frac{1}{n} \#\{0 \leq j < n; i_j \leq q\} \leq \gamma$ and $i_j \leq q$ is equivalent to $f^j(x) \in \mathcal{B}$. \square

To convert this combinatorial result into useful information, we recall the definition of hyperbolic times:

Definition 3.3. We say that $n \in \mathbb{N}$ is an *c-hyperbolic time* for $x \in M$ if

$$\prod_{k=0}^{j-1} \|Df(f^{n-k}(x))^{-1}\| \leq e^{-2cj} \quad \text{for every } 1 \leq j \leq n,$$

and $n \in \mathbb{N}$ is an hyperbolic time for a cylinder $R^n \in \mathcal{R}^n$ if n is a hyperbolic time for all $x \in R^n$. We denote by \mathcal{R}_h^n the set of cylinder $R^n \in \mathcal{R}^n$ with n as an hyperbolic time.

Hyperbolic times are interesting objects because of the following key properties:

Lemma 3.4. *There exists $\delta > 0$ depending only on f and c , such that given any hyperbolic time $n \geq 1$ for a point $x \in M$, and given any $1 \leq j \leq n$, there is an inverse branch $f_{x,n}^{-j}$ of f^j defined on the whole ball of radius δ around $f^n(x)$, which sends $f^n(x)$ to $f^{n-j}(x)$ and satisfies*

$$d(f_{x,n}^{-j}(z), f_{x,n}^{-j}(w)) \leq e^{-jc} d(z, w)$$

for every z, w in the ball $B(f^n(x), \delta)$.

Proof. See [ABV]. \square

Corollary 3.5. *There exists $C > 0$ depending on \mathcal{R} and $\delta > 0$ such that if $R^n \in \mathcal{R}_h^n$, then*

$$d(f^{n-j}(x), f^{n-j}(y)) \leq Ce^{-jc} d(f^n(x), f^n(y)),$$

for every $x, y \in R^n$.

Proof. It suffices to cover the atoms R_1, \dots, R_{p+q} with a finite number of balls of radius δ and the claim follows from a direct application of the lemma 3.4 and the fact that the inner diameter of each R_i is finite. \square

Corollary 3.6. *Given any α -Hölder continuous function $\phi : M \rightarrow M$, there exists a constant A such that for all $R^n \in \mathcal{R}_h^n$ and $x, y \in R^n$ holds*

$$|S_n \phi(x) - S_n \phi(y)| \leq A d(f^n(x), f^n(y))^\alpha.$$

Proof. Since ϕ is α -Hölder continuous,

$$|S_n \phi(x) - S_n \phi(y)| \leq \sum_{j=0}^{n-1} |\phi(f^j(x)) - \phi(f^j(y))| \leq \|\phi\|_\alpha \sum_{j=0}^{n-1} d(f^j(x), f^j(y))^\alpha.$$

By the corollary 3.5 we have $d(f^{n-j}(x), f^{n-j}(y)) \leq Ce^{-jc} d(f^n(x), f^n(y))$ for every $j = 0, \dots, n-1$. These two facts together implies

$$|S_n \phi(x) - S_n \phi(y)| \leq \|\phi\|_\alpha C^\alpha \sum_{j=0}^{n-1} e^{-jc\alpha} d(f^n(x), f^n(y))^\alpha \leq A d(f^n(x), f^n(y))^\alpha.$$

\square

Corollary 3.7. *There exists K such that if $R^n \in \mathcal{R}_h^n$ and $x, y \in R^n$ then*

$$K^{-1} \leq \frac{J_\nu f^n(x)}{J_\nu f^n(y)} \leq K.$$

Proof. Notice that $J_\nu f^n(x) = \lambda^n e^{-S_n \phi(x)}$. Therefore, the corollary 3.6 implies

$$\frac{J_\nu f^n(x)}{J_\nu f^n(y)} = e^{S_n \phi(y) - S_n \phi(x)} \leq e^{A d(f^n(x), f^n(y))^\alpha},$$

which gives the claimed result for $K = e^{AD^\alpha}$, where D is the supremum of the diameters of the rectangles of \mathcal{R} . \square

One of the main features of the measures considered here is contained in the definition in the sequel:

Definition 3.8. We say that a measure η is *expanding* if $\eta(H) = 1$, where

$$H := \left\{ x \in M : \limsup_{n \rightarrow \infty} \frac{1}{n} \# \{0 \leq j \leq n; R^j(x) \in \mathcal{R}_h^j\} > 0 \right\}.$$

To study the expansiveness properties of measures we need the following classical lemma:

Lemma 3.9 (Pliss [P]). *Given $A \geq c_2 > c_1 > 0$, define $\theta = \frac{c_2 - c_1}{A - c_1}$. If b_1, \dots, b_n are real numbers such that $b_i \leq A$ and*

$$\sum_{i=1}^n b_i \geq c_2 n,$$

then there are $l > \theta n$ integers, say $1 \leq n_1 < \dots < n_l \leq n$, such that for $0 \leq k \leq n_i$:

$$\sum_{j=k+1}^{n_i} b_j \geq c_1(n_i - k).$$

Proof. Define $S(m) = \sum_{j=1}^m (a_j - c_1)$, for $1 \leq m \leq n$ and $S(0) = 0$. Then, let $1 < n_1 < \dots < n_l \leq n$ be the maximal sequence with $S(n_i) \geq S(m)$ for all $0 \leq m \leq n_i$, $i = 1, \dots, l$. Since $S(n) > S(0)$, $l > 0$. On the other hand, the definitions means that

$$\sum_{j=m+1}^{n_i} a_j \geq c_1(n_i - m) \quad \text{for } 0 \leq m < n_i \quad \text{and } i = 1, \dots, l.$$

Therefore, it remains only to show that $l > \theta n$. Observe that the definition of $S(m)$ implies

$$S(n_i - 1) < S(n_{i-1}) \rightarrow S(n_i) < S(n_{i-1}) + A - c_1,$$

for $1 < i \leq l$ and, moreover, $S(n_1) \leq A - c_1$ and $S(n_l) \geq S(n) \geq n(c_2 - c_1)$. Thus,

$$n(c_2 - c_1) \leq S(n_l) = \sum_{i=2}^l (S(n_i) - S(n_{i-1})) + S(n_1) < l(A - c_1).$$

This completes the proof. \square

At this point, the combinatorial result of lemma 3.2 permit us to show that any eigenmeasure of the dual of the transfer operator is expanding:

Lemma 3.10. *Let ν be a probability measure such that $\mathcal{L}_\phi^* \nu = \lambda \nu$. Then, there exists $\theta > 0$ depending only on f and c such that*

$$\nu \left(\left\{ x \in M : \limsup_{n \rightarrow \infty} \frac{1}{n} \# \{ 0 \leq j \leq n; R^j(x) \in \mathcal{R}_h^j \} \geq \theta \right\} \right) = 1.$$

In particular, ν is expanding.

Proof. From the proof of the lemma 3.2, we know that, if we define $A(n)$ as the union of the n -cylinders $R^n = [i_0, \dots, i_{n-1}]$ such that

$$\frac{1}{n} \# \{0 \leq j < n; i_j \leq q\} > \gamma,$$

then the sequence $\nu(A(n))$ decreases exponentially with n . Take any $x \notin A(n)$. Then, if (i_0, \dots, i_{n-1}) is the itinerary of x ,

$$\frac{1}{n} \sum_{j=0}^{n-1} \log \sup_{R_{i_j}} \|Df^{-1}\| \leq \log \left((1 + \delta_0)^\gamma \sigma_1^{-(1-\gamma)} \right) \leq -4c < 0.$$

Taking $A = \sup_{x \in M} -\log \|Df(x)^{-1}\|$, $c_1 = 2c$, $c_2 = 3c$ and $b_j = -\log \sup_{R_{i_j}} \|Df^{-1}\|$ in lemma 3.9, we obtain that there exists $n_1 < \dots < n_l \leq n-1$ with $R^{n_i} \in \mathcal{R}_h^{n_i}$ and $l > \theta n$, where $\theta = \frac{c}{A-2c}$. These two facts finishes the proof. \square

An interesting consequence of a measure η being expanding is:

Lemma 3.11. *Let η be an expanding measure. Then, for any measurable set E and any $\epsilon > 0$, there exists hyperbolic cylinders C_1, \dots, C_k such that*

$$\eta(E \Delta \bigcup_{i=1}^k C_i) < \epsilon.$$

Proof. Take $K_1 \subset E$ and $K_2 \subset E^c$ compact sets such that $\eta(E \Delta K_1) < \epsilon/3$, $\eta(E^c \Delta K_2) < \epsilon/3$ and define $r := \text{dist}(K_1, K_2)$. Note that if n is a hyperbolic time for some cylinder C , then $\text{diam}(C) \leq K e^{-cn}$ (see corollary 3.5). Hence, if $C \in \mathcal{R}_h^n$ and $n > n_0$ (with n_0 large enough), we have $\text{diam}(C) \leq K e^{-cn} \leq r$. Since η is expanding (i.e., $\eta(G) = 1$), we can choose $C_i \in \mathcal{R}_h^{n_i}$ intersecting K_1 with $n_i > n_0$ and

$$\eta(K_1 \Delta \bigcup_{i=1}^k C_i) < \epsilon/3.$$

On the other hand, $C_i \cap K_2 = \emptyset$ because $C_i \cap K_1 \neq \emptyset$, $\text{diam}(C) \leq r$ and $r = \text{dist}(K_1, K_2)$.

Therefore, $\eta(E \Delta \bigcup_{i=1}^k C_i) \leq \eta(E - K_1) + \epsilon/3 + \eta(E^c - K_2) \leq \epsilon$. \square

Another useful consequence of expansiveness is the *weak Gibbs property* for the eigenmeasures of \mathcal{L}_ϕ^* :

Lemma 3.12. $\exists K > 0$ such that if $\mathcal{L}_\phi^* \nu = \lambda \nu$, then for every n and $x \in R^n$ with $R^n \in \mathcal{R}_h^n$,

$$K^{-1} \leq \frac{\nu(R^n)}{\exp(S_n \phi(x) - Pn)} \leq K,$$

where $P = \log \lambda$.

Proof. By the lemma 3.1, $J_\eta f^n = e^{nP - S_n \phi}$. Hence, if $R^n = [i_0, \dots, i_{n-1}]$,

$$\eta(f(R_{i_{n-1}})) = \eta(f^n(R^n)) = \int_{R^n} J_\eta f^n d\eta = \int_{R^n} e^{nP - S_n \phi} d\eta.$$

By the corollary 3.7, there exists K_1 (not depending on n) such that for every $x, y \in R^n$

$$K_1^{-1} J_\eta f^n(y) \leq J_\eta f^n(x) \leq K_1 J_\eta f^n(y).$$

It follows that

$$K_1^{-1} \eta(f(R_{i_{n-1}})) \leq \frac{\eta(R^n)}{e^{S_n \phi - nP}} \leq K_1 \eta(f(R_{i_{n-1}})).$$

So, in order to conclude the proof, it suffices to verify that $\eta(f(R_i)) > 0$ for every i . Indeed, from the previous estimate, we have

$$K^{-1} \leq \frac{\eta(R^n)}{e^{S_n \phi - nP}} \leq K,$$

with $K = K_1 \cdot \max\{\sup_i \eta(f(R_i)), \sup_i \frac{1}{\eta(f(R_i))}\}$.

By the third condition in the hypothesis (H1), we know that there exists some N satisfying $f^N(R_i) = M$ for every i . Since η has a Jacobian and M has total probability, a decomposition of $f(R_i)$ into finitely many pieces where f^{N-1} is injective shows that $\eta(f(R_i)) > 0$, as desired. \square

An immediate corollary of the weak Gibbs property is the equivalence of any two eigenmeasures of \mathcal{L}^* :

Corollary 3.13. *Any two eigenmeasures η_1 and η_2 of \mathcal{L}_ϕ^* with same eigenvalue λ are equivalent.*

Proof. Using lemma 3.12, we have for each $R \in \mathcal{R}_h^n$,

$$K^{-1} e^{S_n \phi(x) - Pn} \leq \nu_i(R) \leq K e^{S_n \phi(x) - Pn},$$

for $i = 1, 2$. This implies that $K^{-2} \nu_2(R) \leq \nu_1(R) \leq K^2 \nu_2(R)$. The proof is completed by applying lemma 3.11 and lemma 3.10. \square

The discussion of this subsection closes the study of the dual \mathcal{L}_ϕ^* of the transfer operator and its eigenmeasures (when the local diffeomorphisms f fits our assumptions). Now, we are going to prove some good spectral properties of the transfer operator \mathcal{L}_ϕ itself.

3.2 Invariant cones for the transfer operator

Standing hypothesis of this subsection. For the proof of the results of the present subsection, we assume only (H1), (H2) and the condition (3).

We define the norm $|||g|||_\alpha := \sup_{R_i} \|g|_{R_i}\|_\alpha$, where

$$\|g\|_\alpha := \sup_{x \neq y} \frac{|g(x) - g(y)|}{d(x, y)^\alpha}.$$

In order to prove the main results, we obtain the following Lasota-Yorke type inequality:

Lemma 3.14.

$$|||\tilde{\mathcal{L}}_\phi(g)|||_\alpha \leq \Theta \cdot |||g|||_\alpha + C \cdot \|g\|_\infty,$$

where $\tilde{\mathcal{L}}_\phi := \frac{1}{\lambda} \cdot \mathcal{L}_\phi$ is the normalized Ruelle-Perron-Frobenius operator and $C > 0$, $0 < \Theta < 1$ are real constants.

Proof. For x, x' in the same atom of the partition $\mathcal{R} = \{R_1, \dots, R_d\}$, we have

$$\frac{|\tilde{\mathcal{L}}_\phi g(x) - \tilde{\mathcal{L}}_\phi g(y)|}{d(x, x')^\alpha} \leq \frac{1}{\lambda} \sum_i \frac{|e^{\phi(y_i)} g(y_i) - e^{\phi(y'_i)} g(y'_i)|}{d(x, x')^\alpha} \leq \Theta \cdot |||g|||_\alpha + C \cdot \|g\|_\infty,$$

where

$$\Theta = \frac{p(x)}{\lambda} e^{\max \phi} \sigma_1^{-\alpha} + \frac{q(x)}{\lambda} e^{\max \phi} (1 + \delta_0)^\alpha$$

and

$$C = \frac{|||e^{\max \phi}|||_\alpha}{\lambda} (p(x) \sigma^{-\alpha} + q(x) (1 + \delta_0)^\alpha).$$

Here $p(x)$ is the number of pre-images of x belonging to $\bigcup_{i=q+1}^{p+q} R_i$ and $q(x)$ is the number of pre-images of x belonging to $\bigcup_{i=1}^q R_i$. This completes the proof because the condition (3) implies $\Theta < 1$. \square

The Lasota-Yorke type inequality above ensures the existence of a strictly \mathcal{L}_ϕ -invariant family of cones of positive Hölder continuous functions⁴ In fact, if we put $\Lambda_L := \{g \in C(M) : g > 0 \text{ and } |||g|||_\alpha \leq L \cdot \inf g\}$, it is not hard to prove that

Corollary 3.15. *There exists some $\sigma < 1$ such that $\tilde{\mathcal{L}}_\phi(\Lambda_L) \subset \Lambda_{\sigma L}$, for any L sufficiently large (depending only on Θ and C).*

Proof. Let $g \in \Lambda_L$. By the lemma 3.14 and the definition of Λ_L ,

$$|||\tilde{\mathcal{L}}_\phi g|||_\alpha \leq \Theta \cdot L \cdot \inf g + C \cdot \|g\|_\infty.$$

Now, using that $\|g\|_\infty \leq \inf g + d\varepsilon^\alpha |||g|||_\alpha$, for any function g , we obtain

$$|||\tilde{\mathcal{L}}_\phi g|||_\alpha \leq (\Theta L + C_1) \inf g,$$

⁴C.M. would like to prof. Carlangelo Liverani for interesting discussions who lead the authors to the final expression for the cones Λ_L used here.

where $C_1 = C + C \cdot d \cdot \varepsilon^\alpha$. However, we see that if L is sufficiently large (e.g., $L > \frac{C_1}{\Theta_0 - \theta}$), then $\Theta L + C_1 < \Theta_0 L$ (for any fixed $\Theta < \Theta_0 < 1$). In particular,

$$|||\tilde{\mathcal{L}}_\phi g|||_\alpha \leq \Theta_0 \cdot L \cdot \inf g.$$

Finally, it is not difficult to see that $\inf g \leq e^{\max \phi} \inf \tilde{\mathcal{L}}_\phi g$. So,

$$|||\tilde{\mathcal{L}}_\phi g|||_\alpha \leq \Theta_0 \cdot e^{\max \phi} \cdot L \cdot \inf \tilde{\mathcal{L}}_\phi g.$$

Taking $\sigma = \Theta_0 \cdot e^{\max \phi}$ and using the condition (3), we conclude the proof. \square

Now, we compute the projective metric of Λ_L (see the appendix II for precise definitions):

Lemma 3.16. *The Λ_L -cone metric Ψ_L is given by $\Psi_L(h, g) = \log \frac{B_L(h, g)}{A_L(h, g)}$, where*

$$A_L(h, g) = \inf_{x \neq y, x, y \in R_i, z \in M} \frac{L|x - y|^\alpha g(z) - (g(x) - g(y))}{L|x - y|^\alpha h(z) - (h(x) - h(y))},$$

and

$$B_L(h, g) = \sup_{x \neq y, x, y \in R_i, z \in M} \frac{L|x - y|^\alpha g(z) - (g(x) - g(y))}{L|x - y|^\alpha h(z) - (h(x) - h(y))}.$$

Proof. By definition, $Ah \preceq g$ iff

$$g(x) - Ah(x) \geq 0 \quad \forall x \in M,$$

and

$$|||g - Ah|||_\alpha \leq L \inf(g - Ah).$$

I.e.,

$$A \leq \min \left\{ \inf_x \frac{g(x)}{h(x)}, \inf_{x \neq y, x, y \in R_i, z \in M} \frac{L|x - y|^\alpha g(z) - (g(x) - g(y))}{L|x - y|^\alpha h(z) - (h(x) - h(y))} \right\}.$$

But, if we choose $x_0 \in M$ such that $\inf_x \frac{g(x)}{h(x)} = \frac{g(x_0)}{h(x_0)}$, then it is not hard to verify

$$\frac{L|x - y|^\alpha g(x_0) - (g(x) - g(x_0))}{L|x - y|^\alpha h(x_0) - (h(x) - h(x_0))} \leq \frac{g(x_0)}{h(x_0)}.$$

Hence,

$$A_L(h, g) = \inf_{x \neq y, x, y \in R_i, z \in M} \frac{L|x - y|^\alpha g(z) - (g(x) - g(y))}{L|x - y|^\alpha h(z) - (h(x) - h(y))}.$$

In a similar way,

$$B_L(h, g) = \sup_{x \neq y, x, y \in R_i, z \in M} \frac{L|x - y|^\alpha g(z) - (g(x) - g(y))}{L|x - y|^\alpha h(z) - (h(x) - h(y))}.$$

\square

The explicit formula of the projective metric of Λ_L allow us to prove that:

Proposition 3.17. $\Lambda_{\sigma L}$ has finite diameter in the Λ_L -cone metric Ψ_L :

$$\text{diam}_{\Psi_L}(\Lambda_{\sigma L}) \leq \Delta := \Delta(\sigma, L) := 2 \log \left(\frac{1+\sigma}{1-\sigma} \right) + 2 \log(1 + \sigma L \cdot d \cdot \varepsilon^\alpha)$$

Proof. From the definition of Λ_L , we have for any $g \in \Lambda_{\sigma L}$,

$$|||g|||_\alpha \leq \sigma L \inf g.$$

In particular, we get

$$\sup g \leq \inf g + \sigma L \cdot d \cdot \varepsilon^\alpha \inf g.$$

However, the lemma 3.16 says that

$$\Psi_L(h, g) = \log \sup_{x, y \in R_i, u, v \in R_j, z, w \in M} \frac{L \cdot g(z) - \frac{(g(x)-g(y))}{|x-y|^\alpha}}{L \cdot h(z) - \frac{(h(x)-h(y))}{|x-y|^\alpha}} \cdot \frac{L \cdot h(w) - \frac{(h(u)-h(v))}{|u-v|^\alpha}}{L \cdot g(w) - \frac{(g(u)-g(v))}{|u-v|^\alpha}}.$$

Putting these facts together, we obtain

$$\Psi_L(h, g) \leq 2 \log \left(\frac{1+\sigma}{1-\sigma} \right) + 2 \log(1 + \sigma L \cdot d \cdot \varepsilon^\alpha),$$

for any $h, g \in \Lambda_{\sigma L}$. □

From these results, it is quite standard to construct a f -invariant measure which is absolutely continuous with respect to ν whose density is Hölder continuous:

Theorem 3.18. *There exists an unique f -invariant probability μ absolutely continuous with respect to ν whose Radon-Nikodym derivative h is positive and Hölder continuous.*

Proof. Before starting the proof, we show a simple lemma relating some Banach norms and the partial orderings induced by the cones Λ_L :

Lemma 3.19. *For all $\varphi, \psi \in \Lambda_L$,*

$$-\varphi \preceq \psi \preceq \varphi \quad \text{implies} \quad \begin{cases} \int |\psi| d\nu \leq \int |\varphi| d\nu, \\ \sup_M |\psi| \leq \sup_M |\varphi| \text{ and} \\ |||\psi|||_\alpha \leq L \sup_M |\varphi|. \end{cases}$$

Proof of lemma 3.19. The relation $-\varphi \preceq \psi \preceq \varphi$ means that $\varphi \pm \psi \in \Lambda_L$. In particular, $\psi \leq \varphi$ and $-\psi \leq \varphi$. So, $\int |\psi| d\nu \leq \int |\varphi| d\nu$ and $\sup_M |\psi| \leq \sup_M |\varphi|$.

To ensure that $|||\psi|||_\alpha \leq L \sup_M |\varphi|$, we take two points $x, y \in R_i$ in the same rectangle R_i with $\psi(x) > \psi(y)$ and observe that

$$\varphi(y) - \psi(y) \leq (1 + Ld(x, y)^\alpha) \cdot (\varphi(x) - \psi(x))$$

$$\varphi(x) + \psi(x) \leq (1 + Ld(x, y)^\alpha) \cdot (\varphi(y) - \psi(y))$$

since $\varphi \pm \psi \in \Lambda_L$. Combining these inequalities, we obtain

$$\psi(x) - \psi(y) + \varphi(y) + \varphi(x) \leq (1 + Ld(x, y)^\alpha) \cdot (\varphi(y) + \varphi(x) + \psi(y) - \psi(x)).$$

Hence,

$$(2 + Ld(x, y)^\alpha) \cdot (\psi(x) - \psi(y)) \leq Ld(x, y)^\alpha (\varphi(x) + \varphi(y)).$$

Therefore,

$$|||\psi|||_\alpha \leq L \sup_M \varphi.$$

□

Now, the corollary 3.15 says: there exists $\sigma < 1$ such that, for all $L > 0$ sufficiently large, $\tilde{\mathcal{L}}_\phi(\Lambda_L) \subset \Lambda_{\sigma L}$. Applying the theorem 8.1 (see appendix II below) and proposition 3.17, we obtain, for arbitrary $\varphi, \psi \in \Lambda_L$,

$$\begin{aligned} \Psi_L(\tilde{\mathcal{L}}_\phi^n(\varphi), \tilde{\mathcal{L}}_\phi^{k+n}(\psi)) &\leq (\tanh(\Delta/4))^{n-1} \Psi_L(\tilde{\mathcal{L}}_\phi(\varphi), \tilde{\mathcal{L}}_\phi^k(\psi)) \\ &\quad (\tanh(\Delta/4))^{n-1} \Delta, \end{aligned} \tag{8}$$

where $\Delta < \infty$ is the diameter of $\Lambda_{\sigma L}$ with respect to the Λ_L -cone metric. Since $\int \tilde{\mathcal{L}}_\phi^n \varphi d\nu = \int \varphi d\nu$, a simple consequence of the lemma 3.19 (with $\|\cdot\|_{(1)} := \int |\cdot| d\nu$ and $\|\cdot\|_{(2)} := \sup_M |\cdot| + \frac{1}{L} \|\cdot\|_\alpha$) and lemma 8.2 (of appendix II below) is: $\tilde{\mathcal{L}}_\phi^n 1$ is a Cauchy sequence for the supremum norm. Its limit $h = \lim_{n \rightarrow \infty} \tilde{\mathcal{L}}_\phi^n 1$ is a fixed point of $\tilde{\mathcal{L}}_\phi$ and $h \in \Lambda_L$. This implies that $\mu = h\nu$ is an invariant probability. Indeed, for a test function u ,

$$\int u \circ f d\mu = \lambda^{-1} \int \mathcal{L}_\phi(h \cdot u \circ f) d\nu = \lambda^{-1} \int \mathcal{L}_\phi h \cdot u d\nu = \int h \cdot u d\nu = \int u d\mu.$$

The uniqueness part of the statement of the theorem follows by taking a fixed point $\varphi \in \Lambda_L$ with $\int \varphi d\nu = 1$ and using the estimate (8). □

Finally, the Lasota-Yorke type inequality of lemma 3.14 implies that the f -invariant probability μ just obtained has fast decay of correlations:

Theorem 3.20. *μ has exponential decay of correlations for Hölder observables: there exists some constants $0 < \tau < 1$ and $K > 0$ such that, for all $u \in L^1(\nu), v \in C^\alpha(M)$ and $n \geq 0$,*

$$\left| \int_M (u \circ f^n) v d\mu - \int_M u d\mu \int_M v d\mu \right| \leq K(u, v) \cdot \tau^n,$$

and μ satisfies the central limit theorem: for all u Hölder continuous, the random variable

$$\frac{1}{\sqrt{n}} \sum_{j=0}^{n-1} (u \circ f^j - \int_M u d\mu)$$

converges to a Gaussian law. More precisely, let u be a α -Hölder continuous function and

$$\sigma^2 := \int v^2 d\mu + 2 \sum_{j=1}^{\infty} v \cdot (v \circ f^j) d\mu, \quad \text{where} \quad v = u - \int_M u d\mu.$$

Then $\sigma < \infty$ and $\sigma = 0$ iff $u = \varphi \circ f - \varphi$ for some $\varphi \in L^1(\mu)$. If $\sigma > 0$, then, for every interval $A \subset \mathbb{R}$,

$$\mu \left(x \in M : \frac{1}{\sqrt{n}} \sum_{j=0}^{n-1} \left(u(f^j(x)) - \int_M u d\mu \right) \in A \right) \rightarrow \frac{1}{\sigma \sqrt{2\pi}} \int_A e^{-\frac{t^2}{2\sigma^2}} dt,$$

as $n \rightarrow \infty$.

Proof. The plan of the proof is, firstly, to prove the exponential decay of correlations and, then, to use this information to get the central limit theorem.

We note that the correlation function

$$C_{u,v}(n) := \int_M (u \circ f^n) v d\mu - \int_M u d\mu \int_M v d\mu.$$

can be rewritten as

$$C_{u,v}(n) = \int_M (u \circ f^n) v h d\nu - \int_M u d\mu \int_M v h d\nu.$$

Suppose first that $vh \in L$, for some sufficiently large L . It is no restriction to assume that $\int v h d\nu = 1$. Then, denoting $\|u\|_1 := \int |u| d\mu$,

$$\begin{aligned} \left| \int (u \circ f^n) v h d\nu - \int u d\mu \int_M v h d\nu \right| &= \left| \int u \left(\frac{\tilde{\mathcal{L}}_\phi^n(vh)}{h} - 1 \right) d\mu \right| \\ &\leq \left\| \frac{\tilde{\mathcal{L}}_\phi^n(vh)}{h} - 1 \right\|_0 \cdot \|u\|_1. \end{aligned}$$

To estimate the term $\left\| \frac{\tilde{\mathcal{L}}_\phi^n(vh)}{h} - 1 \right\|_0$, we note that $h > 0$ is a (Hölder) continuous function on a compact manifold M ; so, applying the estimate (8) and the lemma 8.2 of appendix II (with $\|\cdot\|_{(1)} := \int |\cdot| d\nu$ and $\|\cdot\|_{(2)} := \sup_M |\cdot| + \frac{1}{L} \|\cdot\|_\alpha$), we have

$$\left\| \frac{\tilde{\mathcal{L}}_\phi^n(vh)}{h} - 1 \right\|_0 \leq (e^{D\tau^n} - 1) \leq K\tau^n,$$

for some constants $0 < \tau < 1$, $D = D(u, v) > 0$ and $K = K(u, v)$.

In particular,

$$\left| \int (u \circ f^n) v h d\nu - \int u d\mu \int_M v h d\nu \right| \leq K(u, v) \tau^n \|u\|_1,$$

if $vh \in \Lambda_L$.

For a general (A, α) -Hölder continuous function v , we write, for $B > 0$, $vh = \varphi$ and

$$\varphi = \varphi_B^+ - \varphi_B^- \quad \text{where} \quad \varphi_B^\pm = \frac{1}{2}(|\varphi| \pm \varphi) + B.$$

Clearly φ_B^\pm are (A, α) -Hölder continuous and $\varphi_B^\pm \geq B$. Taking B sufficiently large, we get $\varphi_B^\pm \in \Lambda_L$, for any sufficiently large fixed L . So, we can apply the previous estimate for φ_B^\pm . Hence, by linearity, the same estimate holds for φ . Note that the constant $K(u, v)$ has the form

$$K(u, v) \leq K_0 \|u\|_1 (\|v\|_1 + \|v\|_\alpha),$$

where K_0 is independent of u, v . In fact, the first part of the proof gives

$$K(\varphi_B^\pm, v) \leq K_1 \|u\|_1 (\|\varphi\|_1 + B) \leq K_2 \|u\|_1 (\|v\|_1 + \|v\|_\alpha),$$

and the claim follows since $K(u, v) \leq K_3(K(u, \varphi_B^+) + K(u, \varphi_B^-))$.⁵

This concludes the proof of the exponential decay of correlations. Now we are going to use this information to show the central limit theorem. But, before performing the argument, we introduce some notation. Let \mathcal{F} be the Borel sigma-algebra of M and $\mathcal{F}_n := f^{-n}(\mathcal{F})$. Recall that a function $\xi : M \rightarrow \mathbb{R}$ is \mathcal{F}_n -measurable iff $\xi = \xi_n \circ f^n$ for some $(\mathcal{F}_n$, i.e., Borel) measurable ξ_n . Note that $\mathcal{F}_i \supset \mathcal{F}_{i+1}$ for every $i \geq 0$. Also, an f -invariant measure μ is *exact* iff the sigma-algebra

$$\mathcal{F}_\infty = \bigcap_{n \geq 0} \mathcal{F}_n$$

is μ -trivial.⁶ Clearly any exact measure is ergodic, since $\chi_A \in \mathcal{F}_\infty$ if A is f -invariant.

Coming back to the proof of the central limit theorem, a direct corollary of the exponential decay of correlations is the exactness of μ :

Lemma 3.21. *μ is exact.*

Proof. Let $\psi \in L^1(\mu)$ be \mathcal{F}_∞ -measurable. By definition, there are ψ_n Borel measurable functions with $\psi = \psi_n \circ f^n$. Obviously, $\|\psi_n\|_1 = \|\psi\|_1 < \infty$.

⁵This technical remark is useful when proving the ergodicity of μ , since it allows us to conclude uniform bounds for the decay of the correlation function when the function u varies on a $L^1(\mu)$ bounded set. See the proof of lemma 3.21 below.

⁶Equivalently, all \mathcal{F}_∞ -measurable functions are constant μ -a.e.

Therefore, our previous discussion concernig the asymptotics of the correlation function allows us to obtain, for any α -Hölder continuous function φ ,

$$\begin{aligned} \left| \int (\psi - \int \psi d\mu) \varphi d\nu \right| &= \left| \int (\psi_n \circ f^n) \varphi - \int \psi d\mu \int \varphi d\nu \right| \\ &\leq K'_0(\varphi) \|\psi_n\|_1 \tau^n = K'_0(\varphi) \|\psi\|_1 \tau^n \rightarrow 0, \end{aligned}$$

as $n \rightarrow \infty$, for some $K'_0(\varphi) > 0$. So, $\int (\psi - \int \psi d\mu) \varphi d\nu = 0$ and, in particular, $\psi = \int \psi d\mu$ almost everywhere. In other words, μ is exact. \square

Let $L^2(\mathcal{F}_n) = \{\xi \in L^2(\mu) : \xi \text{ is } \mathcal{F}_n\text{-measurable}\}$. Observe that $L^2(\mathcal{F}_n) \supset L^2(\mathcal{F}_{n+1})$ for each $n \geq 0$. Given $\varphi \in L^2(\mu)$, we denote $\mathbb{E}(\varphi|\mathcal{F}_n)$ the L^2 -orthogonal projection of φ to $L^2(\mathcal{F}_n)$.

Another easy consequence of the exponential decay of correlations is:

Lemma 3.22. *For every α -Hölder continuous function u with $\int u d\mu = 0$ there is $R_0 = R_0(u)$ such that $\|\mathbb{E}(u|\mathcal{F}_n)\|_2 \leq R_0 \tau^n$ for all $n \geq 0$.*

Proof.

$$\begin{aligned} \|\mathbb{E}(u|\mathcal{F}_n)\|_2 &= \sup \left\{ \int \xi u d\mu : \xi \in L^2(\mathcal{F}_n), \|\xi\|_2 = 1 \right\} \\ &= \sup \left\{ \int (\psi \circ f^n) u h d\nu : \psi \in L^2(\mu), \|\psi\|_2 = 1 \right\} \\ &\leq K'_0(uh) \tau^n, \end{aligned}$$

since $\|\psi\|_1 \leq \|\psi\|_2$ and $\int u d\mu = \int u h d\nu = 0$. \square

Now the proof of the central limit theorem can be derived from the lemmas 3.21, 3.22 and the following abstract lemma (whose proof can be found in [V, p. 28–33]):

Lemma 3.23. *Let (M, \mathcal{F}, μ) be a probability space, $f : M \rightarrow M$ be a measurable map such that μ is f -invariant and ergodic. Consider $u \in L^2(\mu)$ such that $\int u d\mu = 0$ and denote by \mathcal{F}_n the non-increasing sequence of sigma-algebras $\mathcal{F}_n = f^{-n}(\mathcal{F}), n \geq 0$. Assume that*

$$\sum_{n=0}^{\infty} \|\mathbb{E}(u|\mathcal{F}_n)\|_2 < \infty.$$

Then, the number $\sigma \geq 0$ defined by

$$\sigma^2 = \int u^2 d\mu + 2 \sum_{n=1}^{\infty} \int u \cdot (u \circ f^n) d\mu$$

is finite and $\sigma = 0$ iff $u = \varphi \circ f - \varphi$ for some $\varphi \in L^2(\mu)$. On the other hand, if $\sigma > 0$, then, for any interval $A \subset \mathbb{R}$,

$$\mu \left(x \in M : \frac{1}{\sqrt{n}} \sum_{j=0}^{n-1} \left(u(f^j(x)) - \int u d\mu \right) \in A \right) \rightarrow \frac{1}{\sigma \sqrt{2\pi}} \int_A e^{-\frac{t^2}{2\sigma^2}} dt,$$

as $n \rightarrow \infty$.

This completes the proof of the theorem. \square

At this stage, we can prove that the f -invariant measure μ constructed above is an equilibrium state for (f, ϕ) .

4 Existence of the equilibrium states

We denote by $g : M \rightarrow (0, \infty)$ the function

$$g(x) := \lambda^{-1} e^{\phi(x)} \frac{h(x)}{h(f(x))},$$

where $h > 0$ is the eigenfunction of \mathcal{L}_ϕ with eigenvalue λ . In particular, we have, for every $x \in M$,

$$\sum_{f(y)=x} g(y) = \frac{\sum_{f(y)=x} e^{\phi(y)} h(y)}{\lambda h(x)} = \frac{\mathcal{L}_\phi h(x)}{\lambda h(x)} = 1. \quad (9)$$

For later reference, we state the following elementary calculus lemma:

Lemma 4.1. *Let p_i, x_i , $i = 1, \dots, n$ be positive real numbers such that $\sum_{i=1}^n p_i = 1$. Then,*

$$\sum_{i=1}^n p_i \log x_i \leq \log \left(\sum_{i=1}^n p_i x_i \right),$$

and the equality holds if and only if the numbers x_i are all equal.

We are ready to prove the main result of this section, namely the existence of an equilibrium measure:

Proposition 4.2. *μ is an equilibrium state of ϕ , $P = \log \lambda$ is the pressure $P(f, \phi)$. Moreover, if η is any equilibrium state of ϕ , then*

- $h_\eta(f) + \int \log g \, d\eta = 0$;
- $J_\eta f(y) = 1/g(y)$ for η -a.e. $y \in M$,

either in the case of theorem A or in the case of theorem B.

Proof of proposition 4.2 in the context of theorem A. Since μ is expanding, by the lemma 3.11 we know that \mathcal{R} is a generating partition for μ . Kolmogorov-Sinai's theorem implies $h_\mu(f) = h_\mu(f, \mathcal{R})$. The equality $h_\mu(f) + \int \phi d\mu = \log \lambda = P$ follows from the lemma 3.12.

Now let η be an invariant measure such that $h_\eta(f) + \int \phi d\eta \geq \log \lambda$, and hence

$$h_\eta(f) + \int \log g d\eta = h_\eta(f) - \log \lambda + \int (\phi + \log h - \log h \circ f) d\eta \geq 0.$$

Applying Rokhlin's formula (see appendix III, corollary 9.5 below)⁷

$$h_\eta(f) = \int \log J_\eta f,$$

into the previous estimate, we get

$$0 \leq \int \frac{g}{g_\eta} d\eta = \int \sum_{f(y)=x} g_\eta(y) \log \frac{g(y)}{g_\eta(y)} d\eta,$$

where $g_\eta := 1/J_\eta f$. However, in view of (9) and lemma 4.1, we get

$$\sum_{f(y)=x} g_\eta(y) \log \frac{g(y)}{g_\eta(y)} \leq \log \left(\sum_{f(y)=x} g_\eta(y) \frac{g(y)}{g_\eta(y)} \right) = \log \left(\sum_{f(y)=x} g(y) \right) = 0$$

at η -a.e. x . From this discussion, we conclude that $\sum_{f(y)=x} g_\eta(y) \log \frac{g(y)}{g_\eta(y)}$ is a non-positive function with non-negative integral (with respect to η). Hence it vanishes at η -a.e. x . Thus, we have the equality $P(f, \phi) = h_\eta(f) + \int \phi d\eta = \log \lambda$. From the lemma 4.1, we also obtain that the values $\log \frac{g(y)}{g_\eta(y)}$ coincides for all $y \in f^{-1}(x)$. I.e., for η -a.e. $x \in M$, there exists a number $c(x)$ such that

$$\frac{g(y)}{g_\eta(y)} = c(x) \quad \text{for every } y \in f^{-1}(x).$$

Since η is an invariant measure,

$$\sum_{f(y)=x} g_\eta(y) = 1,$$

for η -a.e. x . These two facts together with (9) gives

$$c(x) = \frac{\sum_{f(y)=x} g_\eta(y)}{\sum_{f(y)=x} g(y)} = 1,$$

⁷This is the unique point where the extra hypothesis (H3) of theorem A is used. Indeed, if we know that, *a priori*, Rokhlin's formula is true for the measures η with $h_\eta(f) + \int \phi d\eta \geq \lambda$, then (H3) could be removed.

at η -a.e. x . This implies $g(y) = g_\eta(y)$ for every y on the pre-image of a set of full η measure. By invariance of η , this set also has full η measure. This completes the proof. \square

Proof of proposition 4.2 in the context of theorem B. The proof in this case is very similar to the previous one, except for the class of invariant measures satisfying the Rokhlin's formula. Again, since μ is expanding, by the lemma 3.11 we know that \mathcal{R} is a generating partition for μ . Kolmogorov-Sinai's theorem implies $h_\mu(f) = h_\mu(f, \mathcal{R})$. The equality $h_\mu(f) + \int \phi d\mu = \log \lambda = P$ follows from the lemma 3.12.

Now let η be an ergodic equilibrium measure. In particular, $h_\eta(f) + \int \phi d\eta \geq \lambda$, and hence

$$h_\eta(f) + \int \log g d\eta = h_\eta(f) - \log \lambda + \int (\phi + \log h - \log h \circ f) d\eta \geq 0.$$

Proceeding as before, an application of Rokhlin's formula (see appendix III, corollary 9.6 below)⁸

$$h_\eta(f) = \int \log J_\eta f$$

into the previous estimate still gives

$$\frac{g(y)}{g_\eta(y)} = c(x) \quad \text{for every } y \in f^{-1}(x).$$

Since η is an invariant measure,

$$\sum_{f(y)=x} g_\eta(y) = 1,$$

for η -a.e. x . These two facts together with (9) gives

$$c(x) = \frac{\sum_{f(y)=x} g_\eta(y)}{\sum_{f(y)=x} g(y)} = 1,$$

at η -a.e. x . This implies $g(y) = g_\eta(y)$ for every y on the pre-image of a set of full η measure. By invariance of η , this set also has full η measure. This completes the proof. \square

To end this section, we show that equilibrium states are closely related to the eigenmeasures of \mathcal{L}^* :

Lemma 4.3. *If η is an equilibrium measure of ϕ , then $\mathcal{L}_\phi^*(h^{-1}\eta) = \lambda(h^{-1}\eta)$.*

⁸As before, this is also the unique point where the extra hypothesis (H4) of theorem B is used. In fact, if we know that, *a priori*, Rokhlin's formula is true for ergodic equilibrium measures η , then (H4) could be removed (as well as the restrictions (5) and (6)).

Proof. Given any continuous function ξ ,

$$\int \xi d(\mathcal{L}_\phi^*(h^{-1}\eta)) = \int h^{-1}(x) \mathcal{L}_\phi \xi(x) d\eta = \int \sum_{f(y)=x} e^{\phi(y)} \cdot \frac{\xi(y)}{h(f(y))} d\eta(x).$$

From the definition of g and the proposition 4.2,

$$e^{\phi(y)} \frac{1}{h(f(y))} = \lambda \frac{g(y)}{h(y)} = \lambda \frac{g_\eta(y)}{h(y)}.$$

Combining these two equations, we have

$$\int \xi d(\mathcal{L}_\phi^*(h^{-1}\eta)) = \lambda \int \frac{g_\eta(y)}{h(y)} \xi(y) d\eta(x) = \lambda \int \xi h^{-1} d\eta.$$

Because ξ is arbitrary, it holds $\mathcal{L}_\phi^*(h^{-1}\eta) = \lambda(h^{-1}\eta)$. \square

5 Proof of the theorems A and B

Once the previous results about the properties of the Ruelle-Perron-Frobenius transfer operator are established, it is now an easy matter to put the proposition 4.2, lemma 4.3 and theorem 3.20 together to get the proof of our main theorems.

The proposition 4.2 and theorem 3.20 says that the measure μ constructed in the subsection 3.2 is an equilibrium measure with exponential decay of correlations and the central limit theorem property (in both the contexts of theorem A and B). So, it remains only to prove that μ is the unique equilibrium state of f .

By the lemma 4.3, any ergodic equilibrium measure η satisfies $\mathcal{L}_\phi^*(h^{-1}\eta) = \lambda(h^{-1}\eta)$. Thus, if we denote $\nu_\eta = h^{-1}\eta$, then ν_η is an eigenmeasure of \mathcal{L}_ϕ^* with eigenvalue λ . However, the corollary 3.13 implies that μ and η are equivalent measures, that is, $\eta = \xi\mu$ for some μ -integrable function ξ . Since η and μ are invariant measures, $\eta = f^*\eta = (\xi \circ f)\mu$. It follows that $\xi \circ f = \xi$ and, by the ergodicity of η , we get that ξ is constant. Using that η and μ are probability measures, we obtain that $\xi = 1$, i.e., $\eta = \mu$. So, μ is the unique equilibrium state of f .

This completes the proof of theorems A and B.

6 Final comments

As a matter of fact, the machinery developed in the previous sections can be applied to obtain further nice ergodic properties of the equilibrium states just constructed. To illustrate the power of these tools, we take this last section to do three remarks:

- First, we show that the transfer operator can be approximated by finite rank operator; thus, the transfer operator has the spectral gap property and this produces a new proof of the exponential decay of correlations for the equilibrium states considered here.
- Second, we prove that the estimates of Lasota-Yorke type in lemma 3.14 are robust under small *random perturbations*; consequently, using the work of Baladi and Young, we get that our equilibrium measures are *stochastically stable*.
- Third, we prove that the abundance (positive density) of hyperbolic times for generic points (of the equilibrium states in theorems A, B) and the uniqueness of the equilibrium measures μ_ϕ are sufficient to conclude, along the lines of the work of Young, that μ_ϕ has the *large deviation property*.

Since these three remarks are simple modifications (based in the lemmas proved here) of the standard arguments of the thermodynamical formalism of uniformly expanding maps, we only sketch the proof of the claimed results by pointing out the relevant changes of the well-known arguments of the literature.

6.1 Spectral Gap

The outline of this subsection is: we start with a new construction of the equilibrium measures for the robust classes of maps (and potentials) considered above; in the sequel, we show that these equilibrium measures are exact; finally, we approximate the transfer operator by compact operators. The conclusion is a new proof of the exponential mixing of these equilibrium states.

It is clear that the construction of the equilibrium states depends only on the existence of an eigenfunction h for the transfer operator \mathcal{L}_ϕ with eigenvalue λ . To accomplish this objective, we closely follow the approach of Oliveira and Viana [OV2]: take the sequence of functions

$$g_n(x) = \sum_{f^n(y)=x; R^n(y) \in \mathcal{R}_h^n} e^{S_n \phi(y)}$$

and the sequence of numbers

$$Z_n = \sum_{R^n \in \mathcal{R}_h^n} e^{S_n \phi(R^n)},$$

where $S_n \phi(R^n) := \max\{S_n \phi(y) : y \in R^n\}$. Note that $g_n \leq Z_n$.

Lemma 6.1. *There exists a constant $K_1 > 0$ such that*

$$\lambda^{-n} Z_n \leq K_1 \quad \text{and} \quad K_1^{-1} \leq \frac{1}{n} \sum_{j=0}^{n-1} \lambda^{-j} Z_j \leq K_1,$$

for all $n \geq 0$.

Proof. Since the reference measure ν is a non-lacunary weak Gibbs measure (see lemma 3.12), we know that

$$K^{-1}\nu(R^n) \leq \lambda^{-n}e^{S_n\phi(y)} \leq K\nu(R^n),$$

for every $y \in R^n$, $R^n \in \mathcal{R}_h^n$. So, if H_n is the union of all $R^n \in \mathcal{R}_h^n$, it follows

$$K^{-1}\nu(H_n) \leq \lambda^{-n}Z_n \leq K\nu(B_n)$$

This proves the first part of the lemma $\lambda^{-n}Z_n \leq K$ since ν is a probability measure. For the second part, note that we have

$$K^{-1}\frac{1}{n}\sum_{j=0}^{n-1}\nu(B_j) \leq \frac{1}{n}\sum_{j=0}^{n-1}\lambda^{-j}Z_j.$$

Therefore, it suffices to obtain an uniform lower bound on the left hand side quantity, which can be written as

$$\frac{1}{n}\sum_{j=0}^{n-1}\nu(B_j) = \int \int \chi_{B_j}(x)d\nu(x)dm_n(j),$$

where m_n is the normalized counting measure of $\{1, \dots, n\}$. Because ν generic points have a positive density $\theta > 0$ of hyperbolic times (see lemma 3.10), we have

$$\frac{1}{n}\sum_{j=0}^{n-1}\chi_{B_j}(x) > \theta$$

for ν almost every $x \in M$ and all n large enough (depending on x). Taking n large so that the set X of points $x \in M$ verifying this inequality has ν measure bigger than $1/2$, we can apply Fubini theorem to conclude

$$\frac{1}{n}\sum_{j=0}^{n-1}\nu(B_j) = \int \int \chi_{B_j}(x)dm_n(j)d\nu(x) \geq \int_X \theta d\nu(x) \geq \theta/2 > 0.$$

This completes the proof of the lemma. □

A direct consequence of this lemma is

Corollary 6.2. *The sequence $\lambda^{-n}g_n$ is uniformly bounded.*

Next, we control the Holder norm of the sequence g_n :

Lemma 6.3. *There exists a constant $K_2 > 0$ such that*

$$\left| \frac{g_n(x_1)}{g_n(x_2)} - 1 \right| \leq K_2 d(x_1, x_2)^\alpha$$

for any $x_1, x_2 \in f(R_i)$ with $R_i \in \mathcal{R}$.

Proof. Let R^n be an atom of \mathcal{R}_h^n with $f^n(R^n) = R_i$. Then, x_1 and x_2 have unique f^n pre-images y_1 and y_2 inside R^n . By corollary 3.6, there exists a constant $A > 0$ such that

$$|S_n\phi(y_1) - S_n\phi(y_2)| \leq Ad(x_1, x_2)^\alpha.$$

Then,

$$e^{-Ad(x_1, x_2)^\alpha} \leq \frac{g_n(x_1)}{g_n(x_2)} = \frac{\sum_{R^n} e^{S_n\phi(y_1)}}{\sum_{R^n} e^{S_n\phi(y_2)}} \leq e^{Ad(x_1, x_2)^\alpha}$$

Now since $|e^{\pm Ad(x_1, x_2)^\alpha} - 1| \leq K_2 d(x_1, x_2)^\alpha$ if K_2 is large depending on the constant A and the diameter of M , we proved the lemma. \square

Corollary 6.4. *The sequence $\lambda^{-n}g_n$ is equicontinuous.*

Proof. This is easy from the previous lemma which says that the sequence $\lambda^{-n}g_n$ has uniformly bounded Hölder norm. \square

Applying the theorem of Arzela-Ascoli, we conclude that

$$\frac{1}{n} \sum_{j=0}^{n-1} \lambda^{-j} g_j$$

has some subsequence converging uniformly on M to an Hölder function h . We claim that the function h is an eigenfunction of the transfer operator (with eigenvalue λ):

Lemma 6.5. $\mathcal{L}_\phi h = \lambda h$ and h is bounded away from zero and infinity.

Proof. Since $\lambda^{-n}g_n \leq K_1$ and $\frac{1}{n} \sum_{j=0}^{n-1} \lambda^{-j} g_j \geq K_1^{-1}$, we conclude that $K_1^{-1} \leq h \leq K_1$.

To show that $\mathcal{L}_\phi h = \lambda h$, note that

$$\mathcal{L}_\phi h = \lim_k \frac{1}{n_k} \sum_{j=0}^{n_k-1} \lambda^{-j} (\mathcal{L}_\phi g_j - g_{j+1}) + \frac{\lambda}{n_k} \sum_{j=0}^{n_k-1} \lambda^{-j} g_j - \frac{\lambda}{n_k} + \frac{\lambda^{-n_k} g_{n_k}}{n_k}$$

Observe that $\frac{\lambda^{-n_k} g_{n_k}}{n_k}$ goes to zero. Thus, the lemma is proved if we are able to obtain that

$$\frac{1}{n} \sum_{j=0}^{n-1} \lambda^{-j} (\mathcal{L}_\phi g_j - g_{j+1})$$

converges to zero. To begin with, denote by E_{n+1} the set of cylinders

$$E_{n+1} := \{[i_0, \dots, i_n] \in \mathcal{R}^{n+1}; [i_0, \dots, i_n] \notin \mathcal{R}_h^{n+1} \text{ and } [i_1, \dots, i_n] \in \mathcal{R}_h^n\}.$$

Clearly $\#E_{n+1} \leq \#\{[i_0, \dots, i_n] \in \mathcal{R}^{n+1}; \frac{\#\{j: i_j \leq q\}}{n+1} \geq \gamma\} \leq c_0^{n+1}$ for all large n . Hence,

$$\|\lambda^{-j}(\mathcal{L}_\phi g_j - g_{j+1})\|_\infty \leq \lambda^{-i} \sum_{R \in E_{j+1}} e^{S_j \phi(R)} \leq c_0^{j+1} \lambda^{-j} e^{j \max \phi}$$

for any large j . Since $\lambda > e^{\max \phi + c_0}$,

$$\|\lambda^{-j}(\mathcal{L}_\phi g_j - g_{j+1})\|_\infty \leq \eta^j,$$

where $\eta < 1$ and j large. This completes the proof. \square

Next, we prove that the f -invariant measure $\mu = h\nu$ is mixing:

Lemma 6.6. μ is exact.

Proof. We begin with the following abstract claim:

Claim 6.7. Let X be a compact metric space, μ a measure and $\mathcal{P} = \{P_1, \dots, P_l\}$ a measurable partition of X . Consider a sequence of collections C_m of pairwise disjoint sets such that $\text{diam}(C_m) := \max_{C \in C_m} \text{diam}(C) \rightarrow 0$ and $\mu(X - \bigcup_{C \in C_m} C) \rightarrow 0$

for all $m \geq m_0$ as $m \rightarrow \infty$. Then, there exists $\{E_1^{(m)}, \dots, E_l^{(m)}\}$ satisfying

- Each $E_i^{(m)}$ is the union of atoms of C_m ;
- $\lim_{m \rightarrow \infty} \mu(E_i^{(m)} \Delta P_i) = 0$ for all $1 \leq i \leq l$.

Proof of the claim 6.7. Fix $\varepsilon > 0$ and let K_1, \dots, K_l compact sets with $K_i \subset P_i$ and $\mu(P_i - K_i) < \varepsilon$. Put $\delta := \inf_{i \neq j} d(K_i, K_j)$ and consider m large such that $\text{diam}(C_m) < \delta/2$. We separate the elements $C \in C_m$ into some collections $\mathcal{E}_i^{(m)}$ characterized by $C \in \mathcal{E}_i^{(m)}$ if $C \cap K_i \neq \emptyset$. Note that any $C \in C_m$ intersects at most one of the compact sets K_i (and if it does not intersect one of them, we include C arbitrarily into some $\mathcal{E}_i^{(m)}$). Define $E_i^{(m)} := \bigcup_{C \in \mathcal{E}_i^{(m)}} C$. Then,

$$\begin{aligned} \mu(E_i^{(m)} \Delta P_i) &= \mu(P_i - E_i^{(m)}) + \mu(E_i^{(m)} - P_i) \leq \\ &\leq \mu(P_i - K_i) + \mu(X - \bigcup_{C \in C_m} C) + \mu(X - \bigcup_{j=1}^l K_j) \leq (l+2)\varepsilon, \end{aligned}$$

if m is large enough so that $\mu(X - \bigcup_{C \in C_m} C) < \varepsilon$. This proves the claim. \square

Now, we are ready to show that μ is exact. If μ is not exact, there is $A \in \bigcap_{n \geq 0} f^{-n}(\mathcal{F})$ (where \mathcal{F} is the Borel sigma-algebra) such that $\mu(A) > 0$ and

$\mu(M - A) > 0$. Using the claim 6.7, we have that for all $\varepsilon > 0$, there exists $N(\varepsilon)$ such that

$$\frac{\mu(A \cap R^n)}{\mu(R^n)} \geq 1 - \varepsilon,$$

for some $[i_0, \dots, i_{n-1}] = R^n \in \mathcal{R}_h^n$ whenever $n \geq N(\varepsilon)$ (since the hyperbolic n-cylinders forms a collection of sets fitting the hypothesis of the claim 6.7 by the corollary 3.5 and lemma 3.11).

By definition, $A = f^{-n}(A_n)$ where $A_n \in \mathcal{F}$. Because the distortion on hyperbolic cylinders is bounded (see the corollary 3.7), it holds

$$\frac{\mu(A_n \cap R_{i_0})}{\mu(R_{i_0})} \geq 1 - \varepsilon K.$$

Similarly,

$$\frac{\mu(A_n^c \cap R_{j_0})}{\mu(R_{j_0})} \geq 1 - \varepsilon K.$$

Since the hypothesis (H1) ensures that \mathcal{R} is a transitive partition, there is a N such that $f^N(R_i) = M$ for all $1 \leq i \leq p + q$. In particular, the image Q_i of the rectangle R_i under some inverse branch of f^N is contained in R_1 for all i . Take $\delta < 1/2K$ and ε such that

$$\varepsilon \cdot \frac{K \sup_i \mu(R_i)}{c} < \delta,$$

where $c = \min_i \mu(Q_i)$. So,

$$\frac{\mu(A_n \cap Q_{i_0})}{\mu(Q_{i_0})}, \frac{\mu(A_n^c \cap Q_{j_0})}{\mu(Q_{j_0})} > 1 - \delta.$$

Since $A_n = f^{-N}(B_N)$ and $A_n^c = f^{-N}(B_N^c)$, the bounded distortion result of the corollary 3.7 implies

$$\frac{\mu(B_N \cap R_{i_0})}{\mu(R_{i_0})}, \frac{\mu(B_N^c \cap R_{j_0})}{\mu(R_{j_0})} > 1 - \delta K.$$

Summing up these estimates, we get $1 > 2 - 2\delta K$, a contradiction. \square

To proceed further, we introduce the compact operators T_n given by

$$T_n = \mathcal{L}_\phi^n \circ \pi_n,$$

where $\pi_n g(x) := \frac{1}{\nu(R^n(x))} \int_{R^n(x)} g d\nu$. Note that π_n is well-defined since $\nu(R^n) > 0$ for any $R^n \in \mathcal{R}^n$,⁹ and T_n has finite dimensional rank since \mathcal{R}^n is a finite partition.

In the sequel, we show the following result about the approximation of \mathcal{L}_ϕ by compact operators:

⁹This follows directly from $J_\nu f = \lambda e^{-\phi} \leq C := k e^{\max \phi}$ and $0 < \nu(f(R_{i_{n-1}})) = \nu(f^n(R^n)) = \int_{R^n} J_\nu f^n \leq C^n \nu(R^n)$.

Lemma 6.8. *There are constants $C > 0$ and $0 < \theta < 1$ such that*

$$\frac{1}{\lambda^n} \|\mathcal{L}_\phi^n g - T_n g\|_\infty \leq C\theta^n \|g\|_\alpha$$

and

$$\frac{1}{\lambda^n} \|\mathcal{L}_\phi^n g - T_n g\|_\alpha \leq C\theta^n \|g\|_\alpha.$$

Proof. Given g and $R_i^n \in \mathcal{R}^n$, we fix $z_i \in R_i^n$ with $\pi_n g(z) = g(z_i)$ for all $z \in R_i^n$. Also, given $x \in M$, we denote by y_i the pre-image of x under f^n such that $y_i \in R_i^n$. We have

$$\begin{aligned} \frac{1}{\lambda^n} |\mathcal{L}_\phi^n g(x) - T_n g(x)| &\leq \frac{1}{\lambda^n} \sum_{R_i^n \in \mathcal{R}^n} e^{S_n \phi(y_i)} |g(y_i) - g(z_i)| \leq \\ &\frac{e^{n \max \phi}}{\lambda^n} \left(\sum_{R_i^n \in A(n)} |g(y_i) - g(z_i)| + \sum_{R_i^n \notin A(n)} |g(y_i) - g(z_i)| \right), \end{aligned}$$

where $A(n)$ is the collection of n -cylinders $R^n = [i_0, \dots, i_{n-1}]$ with

$$\frac{1}{n} \#\{0 \leq j < n; i_j \leq q\} > \gamma.$$

It follows that

$$\begin{aligned} \frac{1}{\lambda^n} |\mathcal{L}_\phi^n g(x) - T_n g(x)| &\leq \frac{(e^{(c_0 + \max \phi)n} (1 + \delta_0)^n + k^n e^{(-c + \max \phi)n})}{\lambda^n} \|g\|_\alpha \leq \\ &\leq C\theta^n \|g\|_\alpha, \end{aligned}$$

for some $0 < \theta < 1$ (if δ_0 and $\max \phi$ are sufficiently small). Similarly,

$$\begin{aligned} \frac{1}{\lambda^n} \|\mathcal{L}_\phi^n g - T_n g\|_\alpha &\leq \\ \frac{1}{\lambda^n} ((1 + \delta_0)^n e^{c_0 n} + e^{-cn} k^n) (e^{n \max \phi} + 2e^{n \max \phi} n \max \phi) \|g\|_\alpha &\leq \\ C\theta^n \|g\|_\alpha. \end{aligned}$$

This completes the proof of the lemma. \square

Finally, the spectral gap property of the normalized transfer operator $\widetilde{\mathcal{L}}_\phi := \frac{1}{\lambda} \mathcal{L}_\phi$ (i.e., $\text{spec} \widetilde{\mathcal{L}}_\phi = \{1\} \cup \Sigma_0$ where 1 is a simple eigenvalue and Σ_0 is contained in a disc of radius $\theta < 1$) is a direct corollary of the two previous lemmas. Indeed, the lemma 6.8 implies that $\lambda^{-n} \|\mathcal{L}_\phi^n - T_n\|_\alpha \leq C\theta^n$ for some $\theta < 1$, where T_n are compact operators. Hence, the spectral radius of \mathcal{L}_ϕ is $\theta < 1$ at most. So, we can write $\text{spec}(\widetilde{\mathcal{L}}) = \{1 = \beta_0, \dots, \beta_m\} \cup \Sigma_0$ with Σ_0 contained in a disk of radius $\theta < 1$ and β_i are eigenvalues with norm 1. Now, the lemma 6.6 allow us to prove that $m = 0$, i.e., there are no eigenvalues with norm 1 other than $\beta_0 = 1$. In fact, if there exists some non-zero Hölder function v_i with

$\widetilde{\mathcal{L}}_\phi v_i = \beta_i v_i$ and $|\beta_i| = 1$, then the exactness property of μ (see lemma 6.6) implies

$$\begin{aligned} \int u \widetilde{\mathcal{L}}_\phi^n v_i d\nu &= \int (u \circ f^n) v_i d\nu = \int (u \circ f^n)(v_i/h) d\mu \rightarrow \\ &\rightarrow \int u d\mu \int (v_i/h) d\mu = \int u \left(h \int v_i d\nu \right) d\nu \end{aligned}$$

so that $\beta_i^n v_i = \widetilde{\mathcal{L}}_\phi^n v_i$ converges to $h \int v_i d\nu$. Hence, $\beta_i = 1$ and $v_i = h \int v_i d\nu$. This proves that 1 is the only eigenvalue of $\widetilde{\mathcal{L}}_\phi$ with norm 1 and its eigenspace is one-dimensional. Furthermore, its algebraic multiplicity is one since, otherwise there is u_0 with $\widetilde{\mathcal{L}}_\phi^n u_0 = h + n u_0$, a contradiction with the boundedness of $\|\widetilde{\mathcal{L}}_\phi^n\|_\alpha$. Note that the spectral splitting of $\widetilde{\mathcal{L}}_\phi$ is $\mathbb{R}h \oplus X_0$, where $X_0 := \{u : \int u d\nu = 0\}$.

To close this subsection, we show that the spectral gap property implies that μ is exponentially mixing:

Corollary 6.9. *μ has exponential decay of correlations.*

Proof. Clearly,

$$\int (u \circ f^n) v d\mu - \int u d\mu \int v d\mu = \int u \widetilde{\mathcal{L}}_\phi^n (\pi_0(v \cdot h)) d\nu,$$

where $\pi_0(\varphi) := \varphi - \int \varphi d\nu$ is the spectral projection onto $X_0 = \{u : \int u d\nu = 0\}$. Since

$$\begin{aligned} \left| \int u \widetilde{\mathcal{L}}_\phi^n (\pi_0(v \cdot h)) d\nu \right| &\leq \|\widetilde{\mathcal{L}}_\phi^n (\pi_0(v \cdot h))\|_\infty \int |u| d\nu \\ &\leq \|\widetilde{\mathcal{L}}_\phi^n (\pi_0(v \cdot h))\|_\alpha \int |u| d\nu \leq C \theta^n \|\pi_0(v \cdot h)\|_\alpha \int |u| d\nu \leq \\ &C' \int |u| d\nu \|v\|_\alpha \theta^n = K(u, v) \theta^n. \end{aligned}$$

□

6.2 Stochastic Stability

The objective of this subsection is to prove the stability of the equilibrium measures under small random perturbations of the maps and potentials considered here. By small random perturbations of a fixed map f_0 we mean that a metric space Ω , a transformation $\omega \in \Omega \mapsto f_\omega$ with $f_\omega \rightarrow f_{\omega_0} = f_0$ in the $C^{1+\alpha}$ topology as $\omega \rightarrow \omega_0$ for some $\omega_0 \in \Omega$ and a family of probabilities θ_ε on Ω such that $\text{supp}(\theta_\varepsilon) \rightarrow \{\omega_0\}$ as $\varepsilon \rightarrow 0$ are given. In a similar fashion, we can define random perturbations of the potential ϕ .

To handle this problem, we mainly use the works of Baladi and Young [B], [BY]. Namely, it follows from these papers that the stochastic stability of the equilibrium states for the transformations f_0 and potentials ϕ in our hypothesis is guaranteed if we can prove the fundamental lemma:

Lemma 6.10. *If δ_0 in hypothesis (H1) is sufficiently small, there are constants $C > 0$, $0 < \theta < 1$ and, for each n , some $\varepsilon(n) > 0$ such that for any $0 < \varepsilon < \varepsilon(n)$,*

$$\frac{1}{\lambda^n} \|\mathcal{L}_\varepsilon^n g\|_\alpha \leq C\theta^n \|g\|_\alpha + C\|g\|_\infty.$$

and

$$\frac{1}{\lambda^n} \|\mathcal{L}_\varepsilon^n g - \mathcal{L}^n g\|_\infty \leq C\theta^n (\|g\|_\infty + \|g\|_\alpha).$$

Here $\mathcal{L} = \mathcal{L}_\phi$ is the transfer operator of (f_0, ϕ) and \mathcal{L}_ε is the random version of the transfer operator when the random noise has level ε .¹⁰

Proof. The first part of the lemma follows from a simple modification of the arguments in the lemma 3.14 using that the random noise is small. Also, the second part is quite similar. Indeed, if the random noise is small, for every $x \in M$ and y with $f^n(y) = x$, there exists a unique y_ε close to y such that $f_\varepsilon^n(y_\varepsilon) = x$. Hence,

$$\begin{aligned} \frac{1}{\lambda^n} \|\mathcal{L}_\varepsilon^n g(x) - \mathcal{L}^n g(x)\|_\infty &\leq \frac{1}{\lambda^n} \sum_i |e^{S_n \phi_\varepsilon(y_\varepsilon^{(i)})} g(y_\varepsilon^{(i)}) - e^{S_n \phi(y^{(i)})} g(y^{(i)})| \\ &\leq C\theta^n \cdot \|g\|_\alpha + C\xi^n \cdot \|g\|_\infty, \end{aligned}$$

where $\theta, \xi \in (0, 1)$ satisfies

$$1 > \theta^n > \frac{1}{\lambda^n} e^{(c_0 + \max \phi_\varepsilon)n} (1 + \delta_0(\varepsilon))^n + \frac{1}{\lambda^n} e^{n \max \phi_\varepsilon} e^{-4cn} k^n$$

and

$$1 > \xi^n > \frac{2}{\lambda^n} e^{n \max \phi} n \max \phi$$

with $\sigma_1(\varepsilon)$, $\delta_0(\varepsilon)$ such that $\sigma_1(\varepsilon) \rightarrow \sigma_1$, $\delta_0(\varepsilon) \rightarrow \delta_0$ as $\varepsilon \rightarrow 0$. Of course, the smallness of the random noise $(f_\varepsilon, \phi_\varepsilon)$ was used to get this estimate. This finishes the proof since our hypothesis (besides the facts $f_\varepsilon \rightarrow f$ and $\phi_\varepsilon \rightarrow \phi$ as $\varepsilon \rightarrow 0$) ensures the existence of $\theta, \xi < 1$ as above. \square

Remark 6.11. In connection with the results of Arbieto, Matheus and Oliveira, the stochastic stability theorem proved here implies that the quenched equilibrium states of [AMO] are unique and exponentially mixing.

6.3 Large Deviations

Given a transformation $f : M \rightarrow M$ and an invariant measure μ , we say that (f, μ) has the *large deviations property* if for any continuous observable φ and $\rho > 0$,

$$\limsup_{n \rightarrow \infty} \frac{1}{n} \log \mu \left(\left\{ x \in M : \frac{1}{n} S_n \varphi(x) \notin \left(\int \varphi d\mu - \rho, \int \varphi d\mu + \rho \right) \right\} \right) < 0.$$

The goal of this subsection is to prove the following result:

¹⁰See [B] for the precise definitions.

Theorem 6.12. *The equilibrium states μ of the (f, ϕ) in both theorems A and B have the large deviation property:*

$$\begin{aligned} & \limsup_{n \rightarrow \infty} \frac{1}{n} \log \mu(\{x : \frac{1}{n} S_n \varphi \notin (\int \varphi d\mu - \rho, \int \varphi d\mu + \rho)\}) \leq \\ & \sup\{h_\eta(f) + \int \phi d\eta - P : \eta \text{ is } f\text{-invariant and } |\int \varphi d\eta - \int \varphi d\mu| \geq \rho\} \\ & < 0. \end{aligned}$$

To alleviate the notation during the proof of this theorem, we set

$$B_n := \{x \in M : \frac{1}{n} S_n \varphi(x) \geq \rho\}$$

Since φ is continuous, we can find $E \subset U \subset B_n$ such that U is open and $\nu(B_n - E) < \frac{1}{2n} \nu(B_n)$. Using the lemma 3.11, there is a family $\mathcal{F}_n = \mathcal{E}_1 \cup \dots \cup \mathcal{E}_k$ of hyperbolic cylinders contained in U verifying

$$\nu(E \triangle \bigcup_{F \in \mathcal{F}_n} F) \leq \left(1 - \frac{\theta}{4}\right)^k < \frac{1}{2n} \nu(B_n).$$

Note that each $P \in \mathcal{E}_i$ there is $x \in M$ and h_i an hyperbolic time for x such that $P = R^{h_i}(x)$, where $i = 1, \dots, k$ and $n < h_1 < h_2 < \dots < h_k$. Define \mathcal{C}_n the set of all such pairs (x, h_i) , one for each element of \mathcal{F}_n .

Let

$$\sigma_n = \frac{1}{Z_n} \sum_{(x,l) \in \mathcal{C}_n} e^{S_l \xi(x)} \cdot \delta_x, \text{ where } Z_n = \sum_{(x,l) \in \mathcal{C}_n} e^{S_l \xi(x)} \text{ and } \xi = \phi - P.$$

Using the lemma 9.9 in [W], we obtain

$$H_{\sigma_n}(\bigvee_{i=0}^{h_k-1} f^{-i}(\mathcal{R})) + \int S_{l(x)} \xi(x) d\sigma_n(x) = \log \sum_{(x,l) \in \mathcal{C}_n} e^{S_l \xi(x)},$$

where $l(x)$ is the unique integer l such that $(x, l) \in \mathcal{C}_n$. Since $\max \phi < P$, it follows that $S_{l(x)-n} \xi(x) < 0$ whenever $l(x) > n$. Therefore,

$$H_{\sigma_n}(\bigvee_{i=0}^{h_k-1} f^{-i}(\mathcal{R})) + \int S_n \xi(x) d\sigma_n(x) \geq \log \sum_{(x,l) \in \mathcal{C}_n} e^{S_l \xi(x)}$$

Putting $\eta_n = \frac{1}{n} \sum_{i=0}^n f_*^i \sigma_n$ and η any accumulation point of η_n , we can repeat the arguments in p. 220 of [W] to derive that

$$\limsup_{n \rightarrow \infty} \frac{1}{n} \log Z_n \leq h_\eta(f, \mathcal{R}) + \int \xi d\eta \leq h_\eta(f) + \int \phi d\eta - P.$$

Observe that η_n is a convex combination of measures of the form $\frac{1}{n} \sum_{i=0}^{n-1} \delta_{f^i(x)}$.

In particular,

$$\int \varphi d\eta_n = \frac{1}{Z_n} \sum_{(x,l) \in \mathcal{C}_n} e^{S_l \xi(x)} \cdot \frac{1}{n} \sum_{i=0}^{n-1} \varphi(f^i(x)) \geq \frac{1}{Z_n} \sum_{(x,l) \in \mathcal{C}_n} e^{S_l \xi(x)} \cdot \rho = \rho.$$

So, $\int \varphi d\eta \geq \rho$ since φ is continuous.

Also, the definition of \mathcal{C}_n and the non-lacunary weak Gibbs property of ν (see the lemma 3.12) implies

$$\begin{aligned} \nu(B_n) &\leq \frac{n}{n-1} \nu\left(\bigcup_{F \in \mathcal{F}_n} F\right) \leq \frac{n}{n-1} \sum_{(x,l) \in \mathcal{C}_n} \nu(R^l(x)) \\ &\leq \frac{n}{n-1} \sum_{(x,l) \in \mathcal{C}_n} K e^{S_l \xi(x)} = \frac{Kn}{n-1} Z_n. \end{aligned}$$

Hence, we constructed an invariant measure η with $\int \varphi d\eta \geq \rho$ and

$$\limsup_{n \rightarrow \infty} \frac{1}{n} \log \nu(B_n) \leq \limsup_{n \rightarrow \infty} \frac{1}{n} \log Z_n \leq h_\eta(f) + \int \phi d\eta - P.$$

This estimate allows us to prove the large deviations property for our equilibrium state μ . Indeed, since $\mu = h\nu$ with h (Hölder) continuous, we showed that for any continuous function φ and any fixed number $\rho > 0$,

$$\begin{aligned} \limsup_{n \rightarrow \infty} \frac{1}{n} \log \mu(\{x : \frac{1}{n} S_n \varphi \notin (\int \varphi d\mu - \rho, \int \varphi d\mu + \rho)\}) &\leq \\ \sup\{h_\eta(f) + \int \phi d\eta - P : \eta \text{ is } f\text{-invariant and } |\int \varphi d\eta - \int \varphi d\mu| \geq \rho\}. \end{aligned}$$

Thus, our task is to show that the right-hand side of this estimate is strictly negative. Note that we have shown in the proof of proposition 4.2 that $h_\eta(f) + \int \phi d\eta - P \leq 0$ for any f -invariant measure η . If the supremum in the right side is zero, there exists a sequence of f -invariant measures η_n with $|\int \varphi d\eta_n - \int \varphi d\mu| \geq \rho$ and $h_{\eta_n}(f) + \int \phi d\eta_n - P \rightarrow 0$. The proof of the Rokhlin's formula in the appendix III (see corolaries 9.5 and 9.6 below) says that the η_n are expanding measures for sufficiently large n , so that the lemma 9.4 can be used to give that any partition with small diameter is generating for any such η_n . Taking any accumulation point η of the sequence η_n , then we have $h_{\eta_n}(f) \rightarrow h_\eta(f)$. Therefore, since the potential ϕ is continuous,

$$h_\eta(f) + \int \phi d\eta - P = \lim_{n \rightarrow \infty} h_{\eta_n}(f) + \int \phi d\eta_n - P = 0.$$

In particular, η is an equilibrium measure. By uniqueness, the measures η and μ coincides. But this a contradiction since $\int \varphi d\eta = \lim \int \varphi d\eta_n \notin (\int \varphi d\mu - \rho, \int \varphi d\mu + \rho)$. This completes the proof of the large deviations property of μ .

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7 Appendix I: a combinatorial lemma

Consider the set

$$I_{\gamma,n} := \{(i_0, \dots, i_{n-1}) \in \{1, \dots, q, \dots, q+p\}^n; \#\{0 \leq j < n; i_j \leq q\} > \gamma n\}$$

By definition,

$$\#I_{\gamma,n} \leq \sum_{r \geq \gamma n} \binom{n}{r} p^{n-r} q^r.$$

The general plan of the proof of lemma 2.2 is: our goal is to estimate the rate of exponential growth of $I_{\gamma,n}$ (when $n \rightarrow \infty$) for γ close to 1. A natural way to do this is to use the Stirling’s formula to compare the binomial numbers with exponential functions, in order to get some good asymptotics.

Now, let us see how this strategy works. A direct consequence of Stirling’s formula is the existence of a universal constant $B > 0$ such that:

$$\binom{n}{r} \leq B \left(\left(1 + \frac{1}{k}\right) (1+k)^{1/k} \right)^r \leq B \left((1+k)^{1/k} \cdot \left(1 + \frac{1}{k}\right) \right)^n.$$

for every $r \geq \frac{k}{k+1}n$. So, if $\gamma \geq \frac{k}{k+1}$, then

$$c_\gamma \leq \log \left(\left(1 + \frac{1}{k}\right) (1+k)^{1/k} p^{1/(k+1)} q \right).$$

Since the right-hand side of this inequality goes to $\log q$ when $k \rightarrow \infty$, this completes the proof of lemma 2.2.

8 Appendix II: positive operators and cones

This appendix presents some results of the theory of projective metrics on cones and positive operators (due to Garrett Birkhoff) used in the subsection 3.2.¹¹

¹¹For sake of simplicity, we consider cones and positive operators on Banach spaces only (which is precisely the context of subsection 3.2); however, the arguments can be extended to the general case of vector spaces with a partial ordering satisfying some mild conditions. See [L] for more details.

Let \mathcal{B} be a Banach space with the topology of the norm. A subset $\Lambda \subset \mathcal{B} - \{0\}$ is a *cone* if $r \cdot v \in \Lambda$ for all $v \in \Lambda$ and $r \in \mathbb{R}^+$. The cone Λ is *closed* if $\Lambda \cup \{0\}$ and Λ is *convex* if $v + w \in \Lambda$ for all $v, w \in \Lambda$.

A convex cone Λ with $\Lambda \cap (-\Lambda) = \emptyset$ determine a *partial ordering* \preceq on \mathcal{B} :

$$w \preceq v \text{ iff } v - w \in \Lambda \cup \{0\}.$$

In the sequel, our cones Λ are assumed to be closed, convex and $\Lambda \cap (-\Lambda) = \emptyset$. Given a cone Λ and two vectors $v, w \in \Lambda$, we define $\Psi(v, w) = \Psi_\Lambda(v, w)$ by

$$\Psi(v, w) = \log \frac{B_\Lambda(v, w)}{A_\Lambda(v, w)},$$

where $A_\Lambda(v, w) = \sup\{r \in \mathbb{R}^+ : r \cdot v \preceq w\}$ and $B_\Lambda(v, w) = \inf\{r \in \mathbb{R}^+ : w \preceq r \cdot v\}$. The (pseudo-)metric Ψ is called the *projective metric* of Λ (or Λ -*metric* for brevity).¹²

A key result of this appendix is:

Theorem 8.1. *Let Λ_i be a closed convex cone (with $\Lambda_i \cap (-\Lambda_i) = \emptyset$) in a Banach space \mathcal{B}_i , for $i = 1, 2$. If $\mathcal{L} : \mathcal{B}_1 \rightarrow \mathcal{B}_2$ is a linear operator such that $\mathcal{L}(\Lambda_1) \subset \Lambda_2$, then*

$$\Psi_{\Lambda_2}(\mathcal{L}v, \mathcal{L}w) \leq \tanh \left(\frac{\text{diam}_{\Psi_{\Lambda_2}}(\mathcal{L}\Lambda_1)}{4} \right) \cdot \Psi_{\Lambda_1}(v, w),$$

for any $v, w \in \Lambda$.

Proof. Take $v, w \in \Lambda_1$. If $A_{\Lambda_1}(v, w) = 0$ or $B_{\Lambda_1}(v, w) = \infty$, the desired inequality is easy. On the other hand, if $A := A_{\Lambda_1}(v, w) \neq 0$ and $B := B_{\Lambda_1}(v, w) \neq \infty$, then

$$\Psi_{\Lambda_1}(v, w) = \log \frac{B}{A}$$

and $A \cdot v \preceq w$ and $w \preceq B \cdot v$. Observe that $A_{\Lambda_1}(v, w)$ and $B_{\Lambda_1}(v, w)$ are non-negative. In particular, if $\text{diam}_{\Psi_{\Lambda_2}}(\mathcal{L}\Lambda_1) = \infty$, the desired inequality follows. If $\Delta := \text{diam}_{\Psi_{\Lambda_2}}(\mathcal{L}\Lambda_1) < \infty$, then

$$\Psi_{\Lambda_2}(\mathcal{L}(w - Av), \mathcal{L}(Bv - w)) \leq \Delta.$$

Hence, there are $r, s \geq 0$ such that

$$r \cdot \mathcal{L}(w - Av) \preceq \mathcal{L}(Bv - w)$$

and

$$\mathcal{L}(Bv - w) \preceq s \cdot \mathcal{L}(w - Av)$$

¹²The justification of the name “projective metric” for the pseudo-metric Ψ is: defining the equivalence relation $v \sim w$ iff $w = r \cdot v$ for some $r \in \mathbb{R}^+$, then Ψ induces a metric on the quotient Λ / \sim .

with $\log \frac{s}{r} \leq \Delta$. Thus,

$$\mathcal{L}w \preceq \frac{B+rA}{1+r} \cdot \mathcal{L}v$$

and

$$\frac{sA+B}{1+s} \cdot \mathcal{L}v \preceq \mathcal{L}w$$

Therefore,

$$\begin{aligned} \Psi_{\Lambda_2}(\mathcal{L}v, \mathcal{L}w) &\leq \log \frac{(B+rA)(1+s)}{(sA+B)(1+r)} = \log \frac{r + e^{\Psi_{\Lambda_1}(v,w)}}{s + e^{\Psi_{\Lambda_1}(v,w)}} - \log \frac{1+r}{1+s} \\ &= \int_0^{\Psi_{\Lambda_1}(v,w)} \frac{(s-r)e^t}{(r+e^t)(s+e^t)} dt \leq \Psi_{\Lambda_1}(v,w) \frac{1 - \frac{r}{s}}{(1 + \sqrt{\frac{r}{s}})^2} \\ &\leq \tanh\left(\frac{\Delta}{4}\right) \Psi_{\Lambda_1}(v,w). \end{aligned}$$

This finishes the proof. \square

Lemma 8.2. *Let Λ be a closed convex cone (with $\Lambda \cap (-\Lambda) = \emptyset$) in a Banach space \mathcal{B} endowed with two norms $\|\cdot\|_{(i)}$, $i = 1, 2$ (not necessarily equivalent), and assume that for all $v, w \in \mathcal{B}$,*

$$-v \preceq w \preceq v \quad \text{implies} \quad \|w\|_{(i)} \leq \|v\|_{(i)} \quad \text{for} \quad i = 1, 2.$$

Then, for all $v, w \in \Lambda$ with $\|v\|_{(1)} = \|w\|_{(1)}$, we have

$$\|v - w\|_{(2)} \leq (e^{\Psi_{\Lambda}(v,w)} - 1) \|v\|_{(2)}.$$

Proof. $\Psi(v, w) = \log \frac{B}{A}$, where $Av \preceq w \preceq Bv$. In particular, $-w \preceq 0 \preceq Av \preceq w$ and, hence, $A\|v\|_{(1)} \leq \|w\|_{(1)}$. Since $\|v\|_{(1)} = \|w\|_{(1)}$, we have $A \leq 1$. Similarly, it is not hard to obtain $B \geq 1$. Thus,

$$-(B-A)v \preceq (A-1)v \preceq w - v \preceq (B-1)v \preceq (B-A)v$$

As a consequence, we have

$$\|w - v\|_{(2)} \leq (B-A)\|v\|_{(2)} \leq \frac{B-A}{A}\|v\|_{(2)} = (e^{\Psi(v,w)} - 1)\|v\|_{(2)}.$$

This concludes the proof of the lemma. \square

9 Appendix III: Rokhlin's formula

This appendix has a lot of non-trivial intersection with the recent work of Oliveira and Viana [OV1]. In particular, the proposition 9.1 (and the lemmas 9.2, 9.3 used in its proof) and the lemma 9.4 below were borrowed from [OV1].

We start with an abstract criterion concerning Rokhlin's formula for certain measures with *generating partitions*.

Let $f : M \rightarrow M$ be a measurable transformation and μ an invariant probability. Suppose that there exist a finite or countable partition \mathcal{P} such that

- (a) f is locally injective (i.e., f is injective on every atom of \mathcal{P});
- (b) \mathcal{P} is f -generating with respect to μ (i.e., $\text{diam}(\mathcal{P}^n(x)) \rightarrow 0$ for μ almost every x , where $\mathcal{P}^n(x)$ is the atom of x in the partition $\mathcal{P}^n := \bigvee_{j=0}^{n-1} f^{-j}(\mathcal{P})$).

Proposition 9.1. *If μ is an invariant measure satisfying (a) and (b) above, then μ verifies Rokhlin's formula*

$$h_\mu(f) = \int \log J_\mu f.$$

Proof. Let $\mathcal{P}_\infty = \bigvee_{j=0}^{\infty} f^{-j}(\mathcal{P})$ and $\mathcal{Q}_\infty = \bigvee_{j=1}^{\infty} f^{-j}(\mathcal{P})$. Note that $\mathcal{Q}_\infty(x) = f^{-1}(\mathcal{P}_\infty(f(x)))$. On the other hand, \mathcal{P} is generating means that $\mathcal{P}_\infty(x) = \{x\}$, and so

$$\mathcal{Q}_\infty(x) = f^{-1}(f(x)),$$

for μ -almost all x .

Denote by $\mathbb{E}_\mu(\varphi|\mathcal{N})$ the (μ_-) conditional expectation of a function $\varphi : M \rightarrow \mathbb{R}$ relative to a partition \mathcal{N} , i.e., the essentially unique \mathcal{N} -measurable function $\mathbb{E}_\mu(\varphi|\mathcal{N})$ such that

$$\int_B \mathbb{E}_\mu(\varphi|\mathcal{N}) d\mu = \int_B \varphi d\mu,$$

for every \mathcal{N} -measurable set B .

At this point we prove the following lemma:

Lemma 9.2. $\mathbb{E}_\mu(\varphi|\mathcal{Q}_\infty) = \sum_{y \in \mathcal{Q}_\infty(x)} \frac{1}{J_\mu f(y)} \varphi(y)$ for μ -a.e. x .

Proof of the lemma 9.2. It is clear that the right-hand side is \mathcal{Q}_∞ -measurable. Let B be any \mathcal{Q}_∞ -measurable set, that is, B consists of entire atoms of \mathcal{Q}_∞ . From the definitions, there exists a measurable set C such that $B = f^{-1}(C)$. Hence, the f -invariance of μ implies

$$\begin{aligned} \int_B \sum_{y \in \mathcal{Q}_\infty(x)} \frac{1}{J_\mu f(y)} \varphi(y) d\mu(x) &= \int_C \sum_{y \in f^{-1}(z)} \frac{1}{J_\mu f(y)} \varphi(y) d\mu(z) \\ &= \sum_{A \in \mathcal{P}} \int_{C_A} \frac{1}{J_\mu f(y_A)} \varphi(y_A) d\mu(z), \end{aligned}$$

where $C_A = f(B \cap A)$ and $y_A = (f|_A)^{-1}(z)$. Since every $f|_A$ is injective, we can use the definition of Jacobian to rewrite the later expression as

$$\sum_{A \in \mathcal{P}} \int_{B \cap A} \varphi(y) d\mu(y) = \int_B \varphi d\mu.$$

This proves the lemma. \square

Recall that the conditional entropy of the partition \mathcal{P} with respect to the partition \mathcal{N} is defined as

$$H_\mu(\mathcal{P}|\mathcal{N}) := \int \sum_{A \in \mathcal{P}} -\mathbb{E}_\mu(\chi_A|\mathcal{N}) \log \mathbb{E}_\mu(\chi_A|\mathcal{N}) d\mu.$$

See [W, definition 4.8]. Using the previous lemma, we can calculate the conditional entropy of \mathcal{P} with respect to \mathcal{Q}_∞ in terms of the Jacobian of μ as follows:

Lemma 9.3.

$$H_\mu(\mathcal{P}|\mathcal{Q}_\infty) = \int \log J_\mu f d\mu.$$

Proof of lemma 9.3. The lemma 9.2 says that

$$\mathbb{E}_\mu(\chi_A|\mathcal{Q}_\infty) = \psi_A \circ f, \quad \text{where} \quad \psi_A(z) = \sum_{y \in f^{-1}(z)} \frac{1}{J_\mu f(y)} \chi_A(y).$$

Observe that if $z \in f(A)$, then $\psi_A(z) = 1/J_\mu f(y_A)$, where $y_A = (f|_A)^{-1}(z)$ and if $z \notin f(A)$, then $\psi_A(z) = 0$. In particular,

$$\begin{aligned} H_\mu(\mathcal{P}|\mathcal{Q}_\infty) &= \int \sum_{A \in \mathcal{P}} -\psi_A(z) \log \psi_A(z) d\mu(z) \\ &= \sum_{A \in \mathcal{P}} \int_{f(A)} \frac{1}{J_\mu f(y_A)} \log J_\mu f(y_A) d\mu(z). \end{aligned}$$

Since f is injective on A , the definition of Jacobian implies

$$H_\mu(\mathcal{P}|\mathcal{Q}_\infty) = \sum_{A \in \mathcal{P}} \int_A \log J_\mu f(y) d\mu(y) = \int \log J_\mu f(y) d\mu(y).$$

This concludes the proof. \square

Now, it is an easy matter to complete the proof of the proposition 9.1. Indeed, since \mathcal{P} is generating, $h_\mu(f) = h_\mu(f, \mathcal{P})$. On the other hand, it is well-known that $h_\mu(f, \mathcal{P}) = H_\mu(\mathcal{P}|\mathcal{Q}_\infty)$ (see [W, Theorem 4.14]). Therefore, the lemma 9.3 allow us to conclude $h_\mu(f) = \int \log J_\mu f d\mu$. \square

For later use, we show the following abstract lemma relating positive Lyapunov exponents and hyperbolic times:

Lemma 9.4. *Given any ergodic measure η whose Lyapounov exponents are all bigger than $8c$ then there exists some $N \in \mathbb{N}$ such that f^N has infinitely many c -hyperbolic times for η almost every point.*

Proof. Since the Lyapounov exponents of η are all bigger than $8c$, for almost every $x \in M$, there exists $n_0(x) \geq 1$ such that

$$\|Df^n(x)w\| \geq e^{6cn}\|w\|, \quad \text{for all } w \in T_x M \text{ and } n \geq n_0(x).$$

In other words, $\|Df^n(x)^{-1}\| \leq e^{-6cn}$ for every $n \geq n_0(x)$. Define $\alpha_n = \eta(\{x : n_0(x) > n\})$. Because f is a local diffeomorphism, we may also fix $K > 0$ such that $\|Df(x)^{-1}\| \leq K$ for all $x \in M$. Then,

$$\int_M \log \|Df^n(x)\| d\eta \leq -6cn + Kn\alpha_n = -(6c - K\alpha_n)n.$$

Since $\alpha_n \rightarrow 0$ when $n \rightarrow \infty$, by choosing N large enough, we guarantee that

$$\int_M \frac{1}{N} \log \|Df^N(x)^{-1}\| d\eta < -4c < 0.$$

Hence, the ergodicity of η implies

$$\lim_{n \rightarrow \infty} \frac{1}{n} \sum_{j=0}^{n-1} \frac{1}{N} \log \|Df^N(f^{Nj}(x))^{-1}\| = \int_M \frac{1}{N} \log \|Df^N(x)^{-1}\| d\eta < -4c.$$

Now, the proof of the lemma is completed by taking $A = \sup_{x \in M} -\log \|Df(x)^{-1}\|$,

$c_1 = 2c$, $c_2 = 3c$ and $b_j = -\log \|Df^N(f^{Nj}(x))^{-1}\|$ in lemma 3.9. \square

Corollary 9.5. *Under the hypothesis of theorem A, any measure η with $h_\eta(f) + \int \phi d\eta \geq \log \lambda$ satisfies Rokhlin's formula:*

$$h_\eta(f) = \int J_\eta f d\eta.$$

Proof. The proposition 9.1 says that it suffices to prove that any η with $h_\eta(f) + \int \phi d\eta \geq \log \lambda$ admits a f -generating partition. Without loss of generality, we can assume that η is ergodic.

The first step is to show that the Lyapounov exponents $\lambda_1(\eta) \geq \dots \geq \lambda_d(\eta)$ of η are positive.

We claim that $\lambda^d(\eta) \geq \log k - \max_{x \in M} \log \|\Lambda^{d-1} Df(x)\| - \epsilon_0(f) := \kappa_0$ whenever $h_\eta(f) + \int \phi d\eta \geq \log \lambda$. Indeed, this is an immediate consequence of Ruelle's inequality:

$$\log k \leq \log \lambda \leq h_\eta(f) + \int \phi d\eta \leq \lambda^d(\eta) + \max_{x \in M} \log \|\Lambda^{d-1} Df(x)\| + \epsilon_0(f)$$

Finally, the conclusion of the proof of the corollary 9.5 is: we know that the Lyapounov exponents of any ergodic measure η are all bigger than κ_0 ; so, the

lemma 9.4 above means that η -almost every point has infinitely many hyperbolic times with respect to f^N , for some $N \in \mathbb{N}$.

Hence, we can use the lemma 3.4 to obtain that any partition \mathcal{P} with diameter sufficiently small is generating. This completes the proof. \square

Corollary 9.6. *Under the hypothesis of theorem B, any equilibrium measure η satisfies Rokhlin's formula:*

$$h_\eta(f) = \int J_\eta f d\eta.$$

Proof. We give two proofs of this result.

The first proof is based on the positivity of all Lyapounov exponents of any ergodic equilibrium state η (see theorem 2.3). So, the abstract lemma 9.4 can be used to conclude the existence of generating partitions, and then, the Rokhlin's formula is a consequence of the abstract proposition 9.1.

The second proof consists in a transference of the problem of Rokhlin's formula for measures on M to the same problem in some subshift of finite type, where this formula is known to be true.¹³ Consider the partition \mathcal{R} and define $\Pi : M \rightarrow \Sigma^+$ a map over a subshift of finite type Σ^+ by

$$\Pi(x) = (i_0, \dots, i_n, \dots) \quad \text{such that } f^n(x) \in R_{i_n}.$$

Observe that this map is an ergodic conjugacy with respect to any measure η such that $[x] = \{x\}$ at η -almost every x . Here, if (i_n) is the itinerary of x (i.e., $f^n(x) \in R_{i_n}$), then

$$[x] := [i_0, \dots, i_n, \dots] := \{y \in M : f^n(y) \in R_{i_n}\}$$

Note that any measure with infinitely many hyperbolic times at almost every point satisfies $[x] = \{x\}$ at generic points. So, any equilibrium measure η can be transported to shift as $\Pi^*\eta$. Therefore, η satisfies Rokhlin's formula, since $\Pi^*\eta$ verifies it (see [BS] for more details). \square

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¹³In fact, although these two proofs are formally different, they are the same in spirit since the proof of Rokhlin's formula for invariant measures in a subshift of finite type goes in the lines of lemma 9.2.

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